Formulation of relativistic spin hydrodynamics based on the entropy-current analysis

Hidetoshi TAYA (Fudan University)

based on arXiv: 1901.06615 [hep-th]

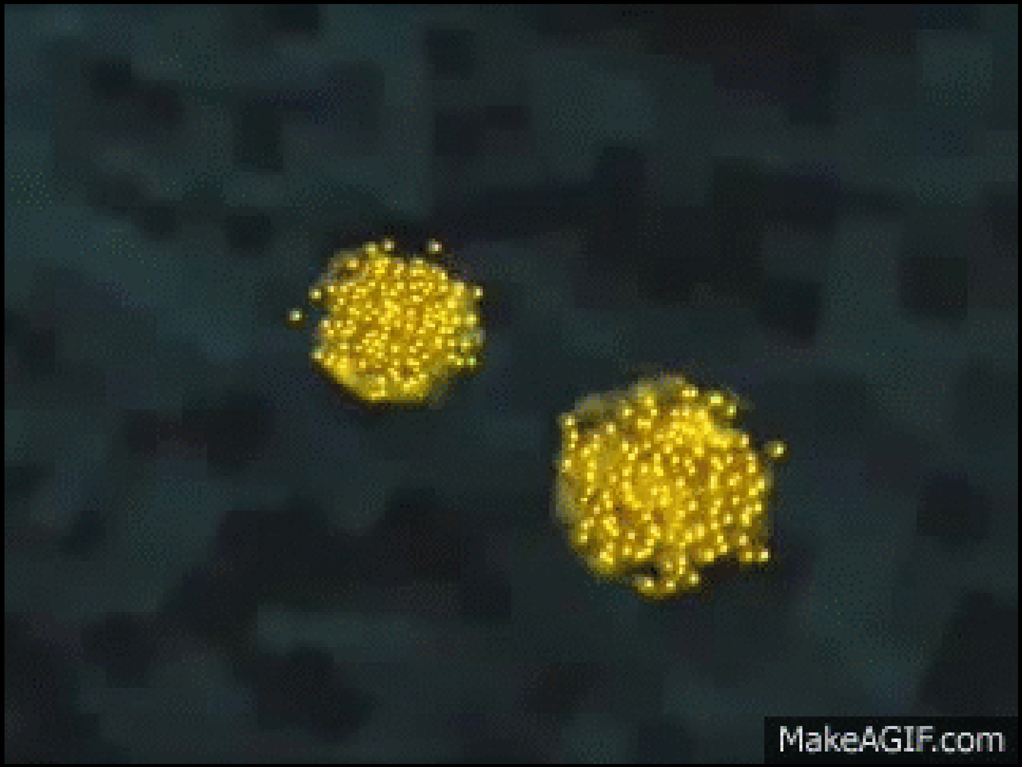
in collaboration with

K. Hattori (YITP), M. Hongo (Keio), X.-G. Huang (Fudan), M. Matsuo (UCAS)

@The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions

2019/04/11

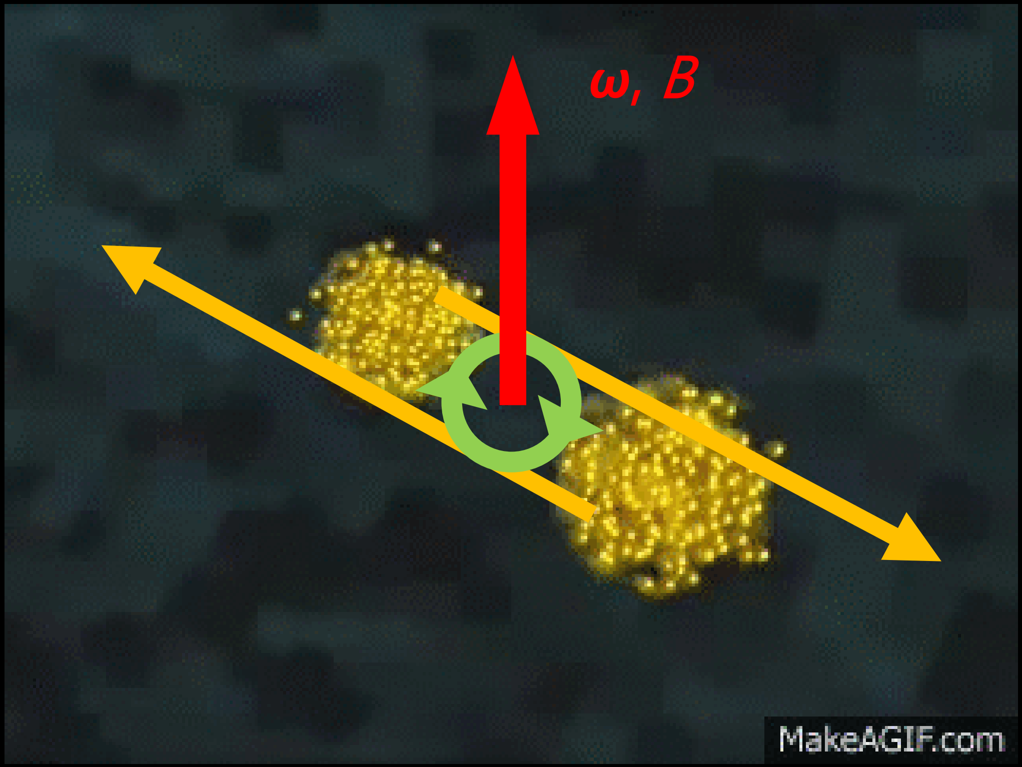
Ultra-relativistic heavy ion collisions



Aim: study quark-gluon plasma (QGP)

Found: QGP behaves like a perfect liquid
and **hydrodynamics works so well**

Huge ω and B



Question: QGP under huge ω and/or B ?

Naïve expectation: QGP is polarized

✓ **Magnetic field *B* effect**

Zeeman splitting (Landau quantization)

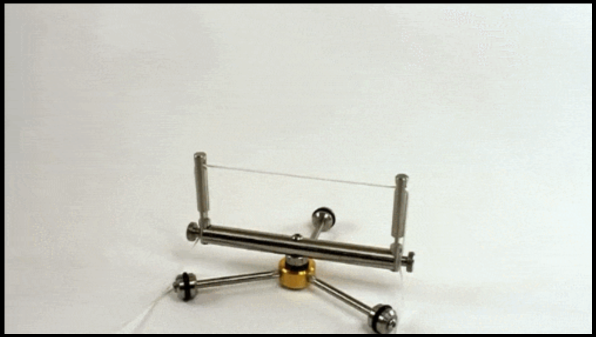
$$E \to E - s \cdot \boldsymbol{q}B$$

➡ charge dependent spin polarization

✓ **Rotation *ω* effect**

Barnett effect

$$E \to E - s \cdot \omega$$



➡ charge *in*dependent spin polarization

Experimental fact

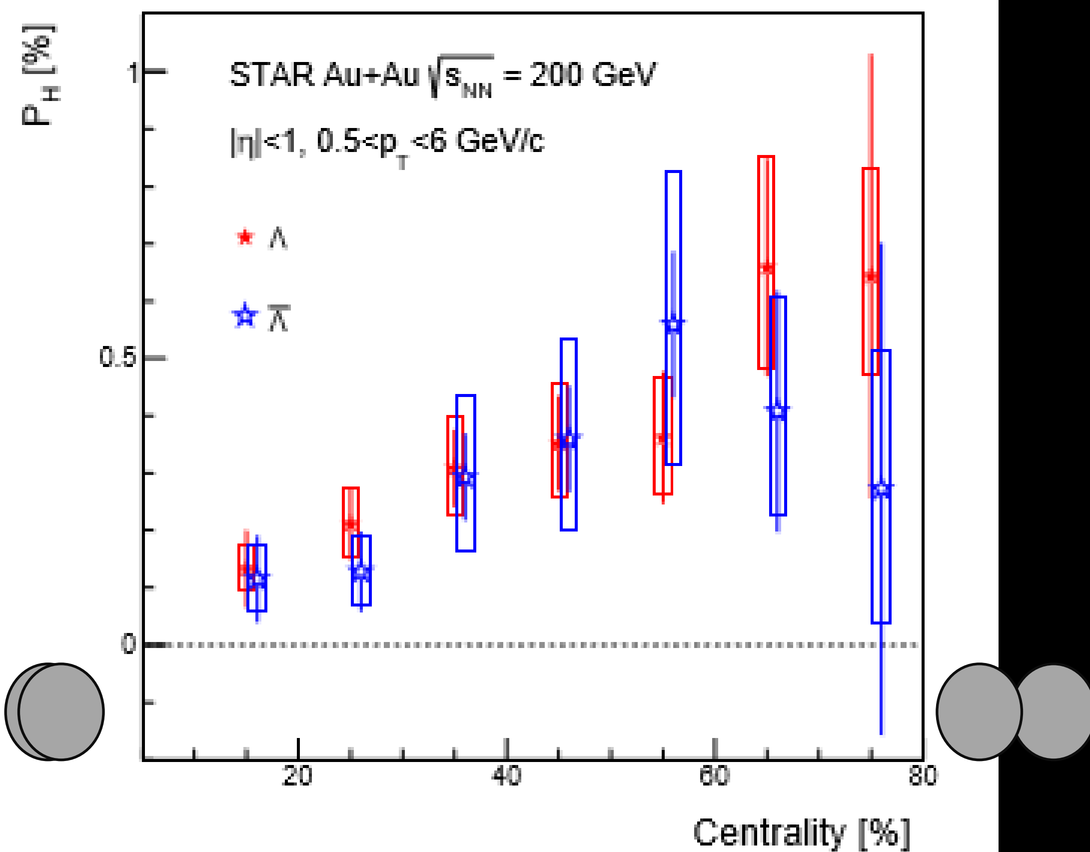


FIG. 5. Λ ($\bar{\Lambda}$) polarization as a function of the collision centrality in Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV. Open boxes and vertical lines show systematic and statistical uncertainties. The data points for $\bar{\Lambda}$ are slightly shifted for visibility.

How about theory?

Hydrodynamics for spin polarized QGP?

Far from complete

Hydrodynamics for spin polarized QGP

✓ **"Hydro simulations" exist, but…**

- usual hydro (i.e., hydro w/o spin) is solved
- thermal vorticity $\varpi^{\mu\nu} \equiv \partial^\mu(u^\nu/T) - \partial^\nu(u^\mu/T)$ is converted into spin via Cooper-Frye formula (???)

✓ **Formulation of relativistic hydrodynamics with spin is still developing**

Current status of formulation of spin hydro

✓ Non-relativistic case

e.g. Eringen (1998); Lukaszewicz (1999)

Already established (e.g. micropolar fluid)

- applied to pheno. and is successful

e.g. spintronics:
Takahashi et al. (2015)

- spin must be dissipative because of mutual conversion between spin and orbital angular momentum

✓ Relativistic case

Some trials exist, but

- only for “ideal” fluid (no dissipative corrections)
- some claim spin should be conserved

Purpose of this talk

- ✓ Formulate **relativistic spin hydrodynamics with 1st order dissipative corrections** for the first time
- ✓ Clarify spin should be **dissipative**

Outline

1. ~~Introduction~~
2. Formulation based on an entropy-current analysis
3. Linear mode analysis
4. Summary

Outline

1. ~~Introduction~~
2. **Formulation based on an entropy-current analysis**
3. Linear mode analysis
4. Summary

Introduction to hydrodynamics w/o spin (1/3)

Hydrodynamics is a low energy effective theory that describes spacetime evolution of IR modes (hydro modes)

textbook by Landau, Lifshitz

✓ Phenomenological formulation (EFT construction)

Step 1: Write down the **conservation law:** $0 = \partial_\mu T^{\mu\nu}$ 4 eqs

Step 2: Express $T^{\mu\nu}$ i.t.o hydro variables (**constitutive relation**)

- define hydro variables: $\{\beta, u^\mu\}$ $(u^2 = -1)$ 1 + (4-1) = 4 DoGs chemical potential for P^μ
- write down all the possible tensor structures of $T^{\mu\nu}$

$$T^{\mu\nu} = f_1(\beta) g^{\mu\nu} + f_2(\beta) u^\mu u^\nu$$
$$+ f_3(\beta)\epsilon^{\mu\nu\rho\sigma}\partial_\rho u_\sigma + f_4(\beta)\partial^\mu u^\nu + f_5(\beta)\partial^\nu u^\mu$$
$$+ f_6(\beta) g^{\mu\nu}\partial^\rho u_\rho + f_7(\beta) u^\mu u^\nu \partial^\rho u_\rho + f_8(\beta) u^\mu \partial_\mu u^\nu + \cdots + O(\partial^2)$$

- simplify the tensor structures by (assumptions in hydro)
  (1) symmetry
  (2) power counting → **gradient expansion**
  (3) other physical requirements → **thermodynamics** *(see next slide)*

✓ **Hydrodynamic eq. = conservation law + constitutive relation**

Introduction to hydrodynamics w/o spin (2/3)

✓ Constraints by thermodynamics

Expand $T^{\mu\nu}$ i.t.o derivatives

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + O(\partial^2)$$

In static equilibrium $T^{\mu\nu} \to T^{\mu\nu}_{(0)} = (e, p, p, p)$, so that

$$T^{\mu\nu}_{(0)} = eu^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$$

1st law of thermodynamics says

$$ds = \beta de, \;\; s = \beta(e+p)$$

By using EoM $0 = \partial_\mu T^{\mu\nu}$, div. of entropy current $S^\mu = su^\mu + O(\partial)$ can be evaluated as

$$\partial_\mu S^\mu = -T^{\mu\nu}_{(1)} \partial_\mu (\beta u_\nu) + O(\partial^3)$$

2st law of thermodynamics requires $\partial_\mu S^\mu \geq 0$, which is guaranteed if **RHS is expressed as a semi-positive bilinear** as

$$-T^{\mu\nu}_{(1)} \partial_\mu (\beta u_\nu) = \sum_{X_i \in T_{(1)}} \lambda_i \, X_i^{\mu\nu} X_{i\,\nu\mu} \geq 0 \text{ with } \lambda_i \geq 0$$

ex) heat current: $2h^{(\mu} u^{\nu)} \equiv h^\mu u^\nu + h^\nu u^\mu \in T^{\mu\nu}_{(1)}$ $(u_\mu h^\mu = 0)$

$$\Rightarrow T^{\mu\nu}_{(1)} \partial_\mu (\beta u_\nu) = -\beta h^\mu \left(\beta \partial_{\perp\mu} \beta^{-1} + u^\nu \partial_\nu u^\mu\right) \geq 0$$

$$\Rightarrow h^\mu = -\kappa \left(\beta \partial_{\perp\mu} \beta^{-1} + u^\nu \partial_\nu u^\mu\right) \text{ with } \kappa \geq 0$$

Introduction to hydrodynamics w/o spin (3/3)

✓ Constitutive relation up to 1st order w/o spin

$$T^{\mu\nu}_{(0)} = eu^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$$

$$T^{\mu\nu}_{(1)} = -2\kappa\left(Du^{(\mu} + \beta\partial_\perp^{(\mu}\beta^{-1}\right)u^{\nu)} - 2\eta\partial_\perp^{<\mu}u^{\nu>} - \zeta(\partial_\mu u^\mu)\Delta^{\mu\nu}$$

heat current — shear viscous effect — bulk viscous effect

✓ Hydrodynamic equation w/o spin

Hydrodynamic eq. = conservation law + constitutive relation

Euler eq.	$0 = \partial_\mu T^{\mu\nu}$	$T^{\mu\nu} = T^{\mu\nu}_{(0)}$
Navier-Stokes eq.	$0 = \partial_\mu T^{\mu\nu}$	$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}$
⋮	⋮	⋮

Formulation of hydrodynamics with spin (1/4)

- ✓ **Strategy is the same**

- ✓ **Phenomenological formulation**

Step 1: Write down the conservation law

Step 2: Construct a constitutive relation
- define hydro variables
- write down all the possible tensor structures
- simplify the tensor structures by e.g. thermodynamics

Formulation of hydrodynamics with spin (2/4)

Step 1: Write down the conservation laws

(1) energy conservation

$$0 = \partial_\mu T^{\mu\nu}$$

(canonical)

(2) total angular momentum conservation

$$\begin{aligned} 0 &= \partial_\mu M^{\mu,\alpha\beta} \\ &= \partial_\mu \left(L^{\mu,\alpha\beta} + \Sigma^{\mu,\alpha\beta}\right) \\ &= \partial_\mu \left(x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + \Sigma^{\mu,\alpha\beta}\right) \end{aligned}$$

$$\psi(x) \to S(\Lambda)\psi(\Lambda^{-1}x)$$

$$\therefore \quad \partial_\mu \Sigma^{\mu,\alpha\beta} = T^{\alpha\beta} - T^{\beta\alpha}$$

- ✓ Spin is **not** conserved if (canonical) $T^{\mu\nu}$ has anti-symmetric part $T^{\mu\nu}_{(\mathrm{a})}$
- ✓ There's **no** a priori reason (canonical) $T^{\mu\nu}$ must be symmetric

Consequence

(1) **Spin must not be a hydro mode** in a strict sense

(2) Nevertheless, it behaves *like* a hydro mode if $T^{\mu\nu}_{(\mathrm{a})} \ll 1$

➡ inclusion of dissipative nature is important

Formulation of hydrodynamics with spin (3/4)

Step 2: Construct a constitutive relation for $T^{\mu\nu}, \Sigma^{\mu,\alpha\beta}$

(1) define hydro variables

4 + 6 = 10 DoGs = # of EoMs

Introduce **spin chemical potential** $\{\beta, u^\mu, \omega^{\mu\nu}\}$ with $\omega^{\mu\nu} = -\omega^{\nu\mu}$

✓ $\{\beta, u^\mu, \omega^{\mu\nu}\}$ are independent w/ each other at this stage ($\omega^{\mu\nu} \neq$thermal vorticity)

(2) simplify the tensor structure by thermodynamics

Expand $T^{\mu\nu}, \Sigma^{\mu,\alpha\beta}$, i.t.o derivatives

$$T^{\mu\nu} = eu^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu) + T^{\mu\nu}_{(1)} + O(\partial^2), \quad \Sigma^{\mu,\alpha\beta} = u^\mu \sigma^{\alpha\beta} + O(\partial^1)$$

where I defined spin density $\boldsymbol{\sigma^{\alpha\beta}}$

Generalize **1st law of thermodynamics with spin** as

$$ds = \beta(de - \omega_{\mu\nu} d\sigma^{\mu\nu}), \;\; s = \beta(e + p - \omega_{\mu\nu}\sigma^{\mu\nu})$$

With EoMs, div. of entropy current $S^\mu = su^\mu + O(\partial)$ can be evaluated as

$$\partial_\mu S^\mu = -T^{\mu\nu}_{(1s)} \frac{\partial_\mu(\beta u_\nu) + \partial_\nu(\beta u_\mu)}{2} - T^{\mu\nu}_{(1a)} \left\{ \frac{\partial_\mu(\beta u_\nu) - \partial_\nu(\beta u_\mu)}{2} - 2\beta\omega_{\mu\nu} \right\} + O(\partial^3)$$

✓ 2nd law of thermodynamics $\partial_\mu S^\mu \geq 0$ gives strong constraint on $T^{\mu\nu}_{(1)}$

✓ In global equilibrium $\partial_\mu S^\mu = 0$, so that ω = thermal vorticity.

Formulation of hydrodynamics with spin (4/4)

✓ Constitutive relation for $T^{\mu\nu}$ up to 1st order with spin

$$T^{\mu\nu}_{(0)} = eu^\mu u^\nu + p(g^{\mu\nu} + u^\mu u^\nu)$$

heat current   shear viscous effect   bulk viscous effect

$$T^{\mu\nu}_{(1)} = -2\kappa\left(Du^{(\mu} + \beta\partial_\perp^{(\mu}\beta^{-1}\right)u^{\nu)} - 2\eta\partial_\perp^{<\mu}u^{\nu>} - \zeta(\partial_\mu u^\mu)\Delta^{\mu\nu}$$

$$-2\lambda\left(-Du^{[\mu} + \beta\partial_\perp^{[\mu}\beta^{-1} + 4u_\rho\omega^{\rho[\mu}\right)u^{\nu]} - 2\gamma\left(\partial_\perp^{[\mu}u^{\nu]} - 2\Delta^\mu_\rho\Delta^\nu_\lambda\omega^{\rho\lambda}\right)$$

"boost heat current"   *"rotational (spinning) viscous effect"*

NEW !

e.g. Eringen (1998); Lukaszewicz (1999)

- ✓ Relativistic generalization of a non-relativistic micropolar fluid
- ✓ "boost heat current" is a relativistic effect

✓ Hydrodynamics equation up to 1st order with spin

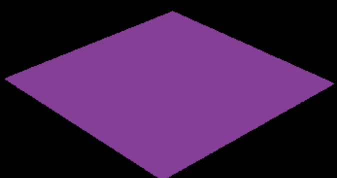
$$0 = \partial_\mu(T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + O(\partial^2)) \qquad \partial_\mu(u^\mu\sigma^{\alpha\beta}) = T^{\alpha\beta}_{(1)} - T^{\beta\alpha}_{(1)} + O(\partial^2)$$

Outline

1. ~~Introduction~~
2. ~~Formulation based on an entropy-current analysis~~
3. **Linear mode analysis**
4. Summary

Linear mode analysis (1/2)

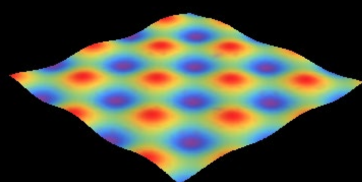
Setup: small perturbations on top of static equilibrium



$\beta = \beta_0$

$u^\mu = (1, \mathbf{0})$

$\omega^{\mu\nu} = 0$



$\beta = \beta_0 + \delta\beta$

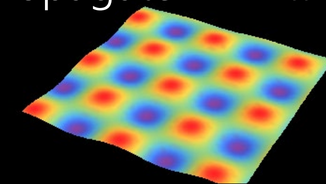
$u^\mu = (1, \mathbf{0}) + \delta u^\mu$

$\omega^{\mu\nu} = 0 + \delta\omega^{\mu\nu}$

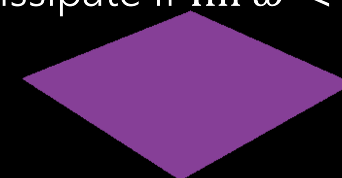
$$i\partial_t \begin{pmatrix} \delta\beta \\ \delta u^\mu \\ \delta\omega^{\mu\nu} \end{pmatrix} = M \begin{pmatrix} \delta\beta \\ \delta u^\mu \\ \delta\omega^{\mu\nu} \end{pmatrix} + O(\delta^2)$$

hydro evolution

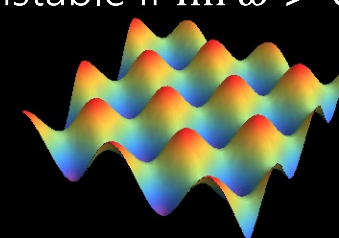
propagate if $\mathrm{Im}\,\omega = 0$



dissipate if $\mathrm{Im}\,\omega < 0$



unstable if $\mathrm{Im}\,\omega > 0$



Linear mode analysis (2/2)

✓ **Hydro w/o spin** $\{\beta, u^\mu\}$

4 modes

2 sound modes $\omega = \pm c_s k + O(k^2)$

2 shear modes $\omega = -i\dfrac{\eta k^2}{e+p} + O(k^4)$

where $c_s^2 \equiv \partial p/\partial e$

✓ **Hydro with spin** $\{\beta, u^\mu, \omega^{\mu\nu}\}$

4 modes

2 sound modes $\omega = \pm c_s k + O(k^2)$

2 shear modes $\omega = -i\dfrac{\eta k^2}{e+p} + O(k^4)$

+ 6 dissipative modes

3 "boost" modes $\omega = -2i\tau_b^{-1} + O(k^2)$

3 "spin" modes $\omega = -2i\tau_s^{-1} + O(k^2)$

where $\tau_s \equiv \dfrac{\partial\sigma^{ij}/\partial\omega^{ij}}{4\gamma}, \tau_b \equiv \dfrac{\partial\sigma^{i0}/\partial\omega^{i0}}{4\lambda}$

✓ We explicitly confirmed that **spin is dissipative**

✓ The time-scale of the dissipation is controlled by the new viscous constants γ, λ

Outline

1. ~~Introduction~~
2. ~~Formulation based on an entropy-current analysis~~
3. ~~Linear mode analysis~~
4. **Summary**

Summary

- ✓ Spin polarization in QGP is one of the hottest topics in HIC. But, its theory, in particular hydrodynamic framework, has not been developed well
- ✓ **Relativistic spin hydrodynamics with 1st order dissipative corrections is formulated** based on the phenomenological entropy-current analysis
- ✓ **Spin must be dissipative** because of the mutual conversion between the orbital angular momentum and spin
- ✓ Linear mode analysis of the spin hydrodynamic equation also suggests that spin must be dissipative

Outlook: extension to 2nd order, Kubo formula, MHD, application to cond-mat, numerical simulations

BACK UP

Linearized hydro eq.

$$M\,\delta\vec{c} = 0$$

where

$$M = \left(\begin{array}{c|ccccccc} A_{3\times3} & & & & O & & & \\ \hline & -i\omega+(\gamma_\perp+\gamma')k_z^2 & +iD_sk_z & 0 & 0 & 0 & 0 & 0 \\ & -2i\gamma'k_z & -i\omega+2D_s & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & -i\omega+(\gamma_\perp+\gamma')k_z^2 & -iD_sk_z & 0 & 0 & 0 \\ O & 0 & 0 & 2i\gamma'k_z & -i\omega+2D_s & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & -i\omega+2D_b & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & -i\omega+2D_b & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & -i\omega+2D_s \end{array}\right)$$

$$A_{3\times3} = \begin{pmatrix} -i\omega+2c_s^2\lambda'k_z^2 & ik_z & -2iD_bk_z \\ ic_s^2k_z & -i\omega+\gamma_\parallel k_z^2 & 0 \\ 2ic_s^2\lambda'k_z & 0 & -i\omega+2D_b \end{pmatrix}$$

and

$$\delta\vec{c} \equiv (\delta\tilde{e}, \delta\tilde{\pi}^z, \delta\tilde{S}^{0z}, \delta\tilde{\pi}^x, \delta\tilde{S}^{zx}, \delta\tilde{\pi}^y, \delta\tilde{S}^{yz}, \delta\tilde{S}^{0x}, \delta\tilde{S}^{0y}, \delta\tilde{S}^{xy})^t$$

Dispersion relations

$$\omega = -2iD_s,$$

$$\omega = -2iD_b,$$

$$\omega = \begin{cases} -2iD_s - i\gamma' k_z^2 + \mathcal{O}(k_z^4), \\ -i\gamma_\perp k_z^2 + O(k_z^4),, \end{cases}$$

$$\omega = \begin{cases} \pm c_s k_z - i\frac{\gamma_\parallel}{2} k_z^2 + \mathcal{O}(k_z^3), \\ -2iD_b - 2ic_s^2 \lambda' k_z^2 + \mathcal{O}(k_z^4). \end{cases}$$

Further simplification by EoM

The 1st order constitutive relation reads

$$\Theta^{\mu\nu}_{(1s)} = 2h^{(\mu}u^{\nu)} + \tau^{\mu\nu}$$
$$\Theta^{\mu\nu}_{(1a)} = 2q^{[\mu}u^{\nu]} + \phi^{\mu\nu}$$

$$h^\mu = -\kappa(Du^\mu + \beta\partial_\perp^\mu T),$$
$$\tau^{\mu\nu} = -2\eta\partial_\perp^{\langle\mu}u^{\nu\rangle} - \zeta\theta\Delta^{\mu\nu},$$
$$q^\mu = -\lambda\big(-Du^\mu + \beta\partial_\perp^\mu T - 4\omega^{\mu\nu}u_\nu\big),$$
$$\phi^{\mu\nu} = -2\gamma\big(\partial_\perp^{[\mu}u^{\nu]} - 2\Delta^\mu_\rho\Delta^\nu_\lambda\omega^{\rho\lambda}\big),$$

By using LO hydro eq.,

$$(e+p)Du^\mu = -\partial_\perp^\mu p + \mathcal{O}(\partial^2)$$

we can further simplify *h,q* as

$$h^\mu = -\kappa\left[\frac{-\partial_\perp^\mu p}{e+p} + \beta\partial_\perp^\mu T + \mathcal{O}(\partial^2)\right] = \mathcal{O}(\partial^2),$$
$$q^\mu = -\lambda\left[\frac{2\partial_\perp^\mu p}{e+p} - 4\omega^{\mu\nu}u_\nu\right] + \mathcal{O}(\partial^2).$$