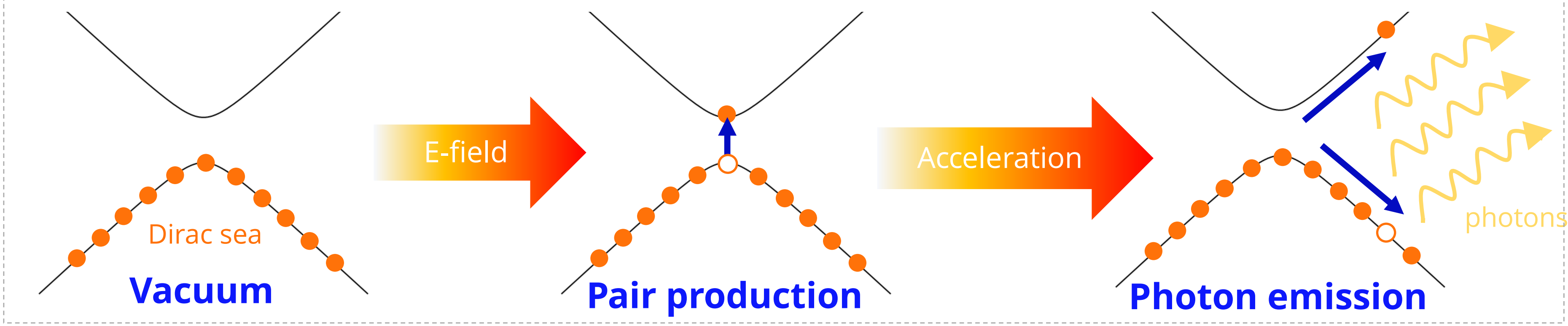
Application of the exact WKB method to particle production and high-harmonics generation (HHG) from the vacuum

Hidetoshi TAYA (RIKEN)

[1] HT, Fujimori, Misumi, Nitta, Sakai, JHEP 03, 082 (2021) [2010.16080]
[2] HT, Hongo, Ikeda, 2105.12446

Problem: HHG from the vacuum

Pair production by E-field leads to emission of photons



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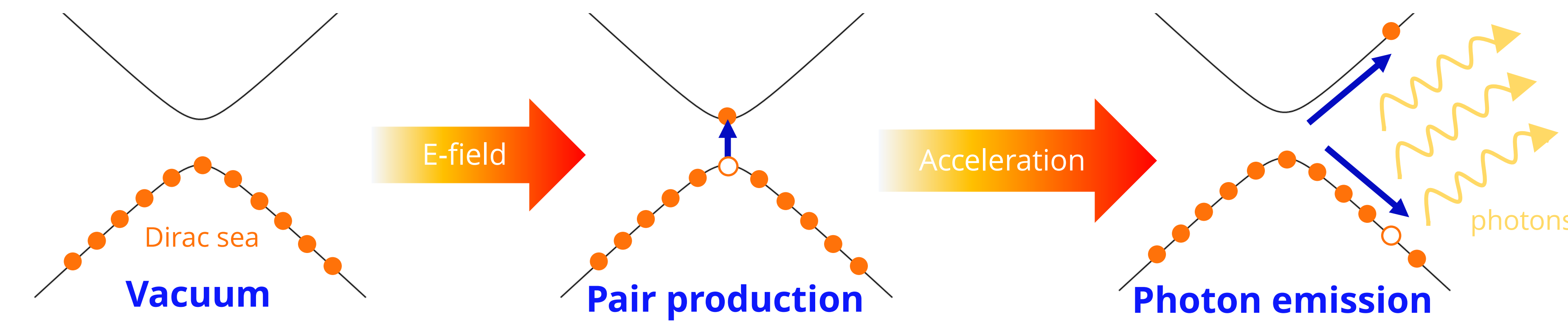

✔ Photons may have high harmonics due to strong-field effects

Suppose E-field has a typical frequency Ω: $\tilde{E}(\omega) \propto \delta(\Omega - \omega)$

- E is weak/fast (large $\gamma_{\mathrm{Keldysh}}$) $\Rightarrow \omega \frac{\mathrm{d}N_\gamma}{\mathrm{d}\omega} = (\text{linear in } \tilde{E}) \Rightarrow$ peaked at $\omega = \mathbf{1} \times \Omega$
- E is strong/slow (small $\gamma_{\mathrm{Keldysh}}$) $\Rightarrow \omega \frac{\mathrm{d}N_\gamma}{\mathrm{d}\omega} = (\text{non} - \text{linear in } \tilde{E}) \Rightarrow \omega = \boldsymbol{n} \times \Omega$ appear

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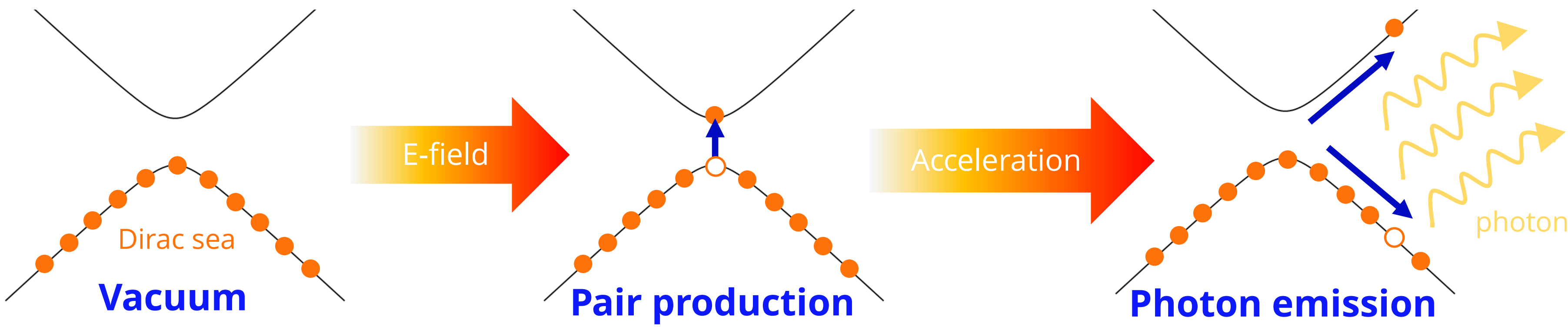

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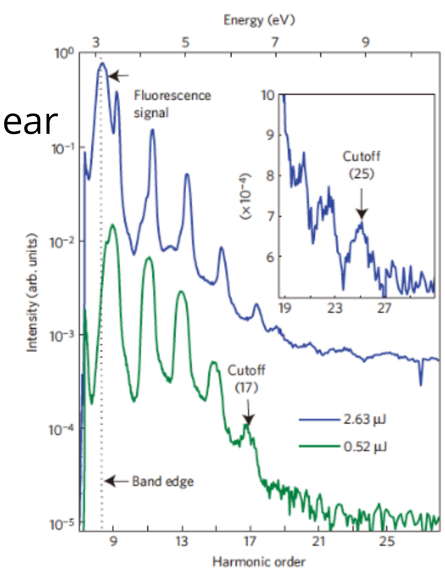
[Ghimire et al., (2011)]



✔ Observation in semi-conductors (also in many other materials)

- Key features: Odd harmonics $n = 1,3,5, \ldots$ & Plateau structure
- Microscopic mechanism of solid-state HHG is still under debate and no established analytical methods
  (e.g., 3 step model of gas-HHG does not necessarily work) [Corkum (1993)]

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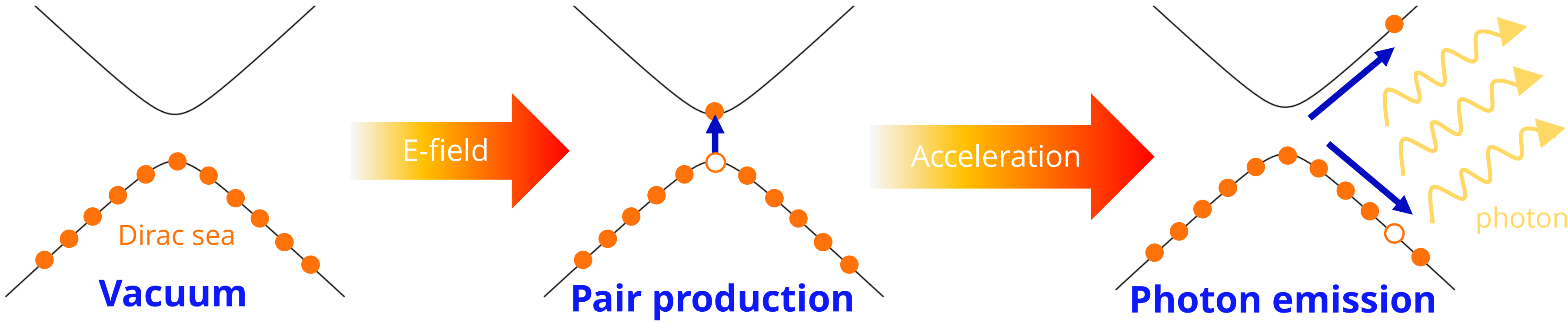

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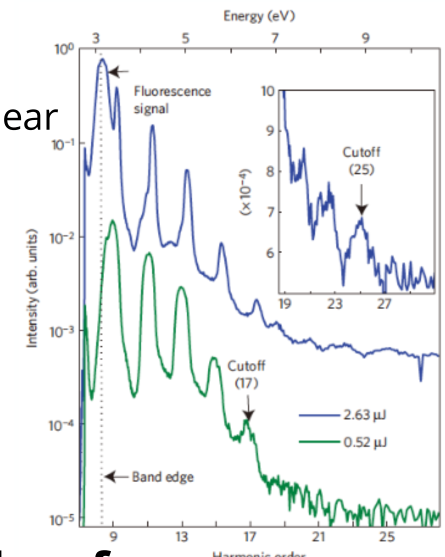

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✔ Natural to expect in QED, but not intensively discussed thus far

- Previous works are based on the Euler-Heisenberg effective action
  $\Rightarrow$ valid only in the static limit $\Omega \to 0 \Rightarrow$ Not satisfactory for $\Omega \neq 0$

[Piazza, Hatsagortsyan, Keitel (2005)]
[Fedotov, Narozhny (2007)]
[Bohl, Ruhl, King (2015)]

What are needed for the formulation ?

① Need to know wavefunction $\psi_\pm$

∵ Photon spectra are given i.t.o. current, which is expressed i.t.o. wavefunctions $\psi_\pm$

$$\omega \frac{\mathrm{d}N}{\mathrm{d}\omega} \sim \left|\omega \tilde{J}(\omega)\right|^2 \quad \text{where} \quad J(t) \coloneqq \left\langle 0;\mathrm{in}\middle|\hat{\bar{\psi}}\gamma^\mu\hat{\psi}\middle|0;\mathrm{in}\right\rangle = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\bar{\psi}_{-,\boldsymbol{p}}\gamma^\mu\psi_{-,\boldsymbol{p}}$$

$$\hat{\psi}(t,\boldsymbol{x}) = \int \mathrm{d}^3\boldsymbol{p}\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{(2\pi)^{3/2}}\left[\psi_{+,\boldsymbol{p}}(t)\hat{a}_{\boldsymbol{p}}^{(\mathrm{in})} + \psi_{-,\boldsymbol{p}}(t)\hat{b}_{-\boldsymbol{p}}^{(\mathrm{in})\dagger}\right] \quad \text{with B.C.} \quad \lim_{t\to-\infty}\psi_{\pm,\boldsymbol{p}} \propto \mathrm{e}^{\mp\mathrm{i}\omega_{\boldsymbol{p}}t} = \text{plane waves}$$

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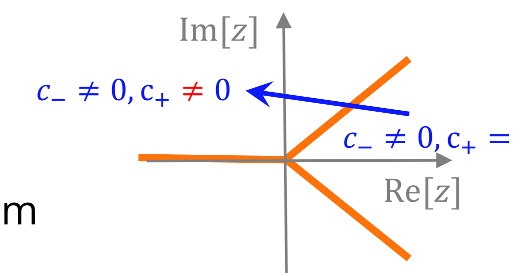
∵ Particle prod. is the origin of HHG and is interpreted as a Stokes phenomenon of ODEs

- **In Math**: Stokes phenomenon ~ Sudden mixing up of positive & negative frequency solutions

Ex.) Airy function: $0 = (\partial_z^2 - z)\mathrm{Ai}(z)$

$$\Rightarrow \mathrm{Ai(z)} \sim c_- \frac{\mathrm{e}^{-2z^{3/2}/3}}{z^{1/4}} + c_+ \frac{\mathrm{e}^{+2z^{3/2}/3}}{z^{1/4}}$$



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$$\hat{\psi} \sim \mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{p}}t}\,\hat{a}_{\mathrm{in}} + \mathrm{e}^{+\mathrm{i}\omega_{\boldsymbol{p}}t}\,\hat{b}^\dagger_{\mathrm{in}} \quad \text{at } t \sim -\infty$$

$$\Rightarrow \hat{\psi} \sim (\alpha\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{p}}t} + \beta\mathrm{e}^{+\mathrm{i}\omega_{\boldsymbol{p}}t})\hat{a}_{\mathrm{in}} + (\alpha^*\mathrm{e}^{+\mathrm{i}\omega_{\boldsymbol{p}}t} + \beta^*\mathrm{e}^{-\mathrm{i}\omega_{\boldsymbol{p}}t})\hat{b}^\dagger_{\mathrm{in}}$$

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whenever Stokes pheno. occur at $t \neq -\infty$

$$\Rightarrow \langle 0;\mathrm{in}|\,\hat{a}^\dagger_t\hat{a}_t\,|0;\mathrm{in}\rangle = \langle 0;\mathrm{in}|\,\hat{b}^\dagger_t\hat{b}_t\,|0;\mathrm{in}\rangle = |\beta|^2 \neq 0$$

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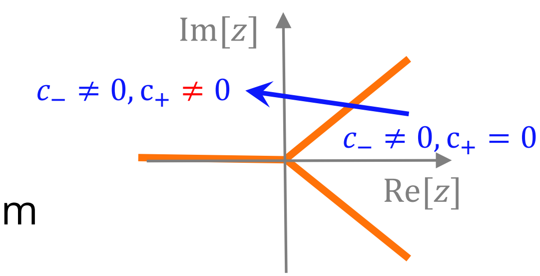

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Any nice analytical method for ① & ② ? ⇒ Exact WKB !

Exact WKB method

[Voros (1983)]
[Pham, Dillinger, Delabaere,
Aoki, Koike, Takei, ...]

[Jeffery (1924)] [Wentzel (1926)]
[Kramers (1926)] [Brillouin (1926)]

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Exact WKB = ”usual” WKB + Borel resum.

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A perturbation theory w.r.t. $\hbar$ (or adiabatic approx.)

- Consider $0 = [\hbar^2 \partial_t^2 + Q(t)]\phi(t) \overset{t \equiv \hbar\tau}{\Leftrightarrow} [\partial_\tau^2 + Q(\hbar\tau)]\phi(\tau)$

$$\Rightarrow \phi_\pm(t;\hbar) := \exp\left[\mp \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}t' \sqrt{Q(t')}\right] \times \sum_{n=0}^{\infty} \psi_{\pm,n}(t)\hbar^n$$

0th order = plane wave
$\sim \exp[\mp \frac{\mathrm{i}}{\hbar}\sqrt{Q}t]$

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Perturbation w.r.t. $\hbar$

- WKB expansion makes sense if the perturbative part is convergent
- However, $\psi_{\pm,n} \sim n!$ in general (e.g., Airy function $Q(t) \propto t$)

$\Rightarrow$ WKB expansion has zero radius of convergence $\Rightarrow$ ill-defined !!!

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A resummation scheme for factorially divergent ~n! series

- Consider the div. part of WKB expansion $\psi_\pm(t;\hbar) := \sum_{n=0}^\infty \psi_{\pm,n}(t)\hbar^n$

① Construct "Borel transformation": $B[\psi_\pm](t;\eta) := \sum_n^\infty \frac{\psi_{\pm,n}(t)}{n!}\eta^n$

② Laplace trans. gives "Borel sum": $\Psi_\pm(t;\hbar) := \int_0^\infty \frac{d\eta}{\hbar}\, e^{-\eta/\hbar} B[\psi_\pm](t;\eta)$

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- $\Psi_\pm$ is well-defined, unless $B[\psi_\pm]$ has singularities on real axis
- Asymptotic expansion of $\Psi_\pm$ = $\psi_\pm$

  $\Rightarrow$ $\Psi_\pm$ is a natural analytic continuation of $\psi_\pm$

  $\Rightarrow$ **$\Psi_\pm$ gives a well-defined version of the WKB solution !**
- **Singularities of $B[\psi_\pm]$ can be used to describe Stokes phenomenon**

Exact-WKB recipe for Stokes phenomenon

Step 1: Draw a Stokes graph

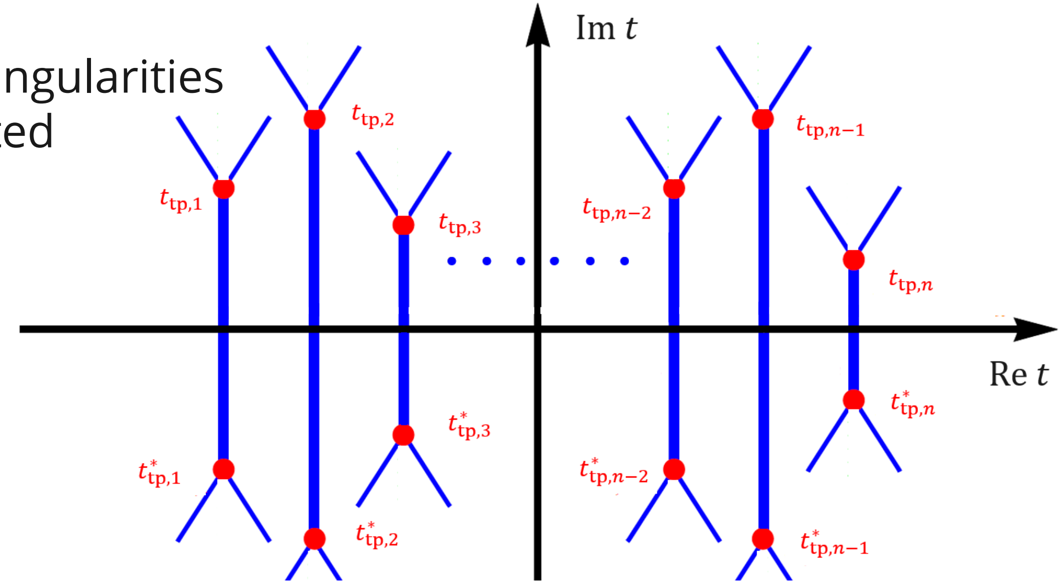
- Laplace trans. of $B[\psi_\pm](t;\eta)$ hits singularities (non-Borel summable) when located on Stokes lines in the t-plane

— : Stokes lines

$$\{t \in \mathbb{C} \mid 0 = \mathrm{Im}\, \mathrm{i} \int_{t_{\mathrm{tp}}}^{t} \mathrm{d}z\, \sqrt{Q(z)}\, \}$$

● : turning points

$$Q(t_{\mathrm{tp}}) = 0$$



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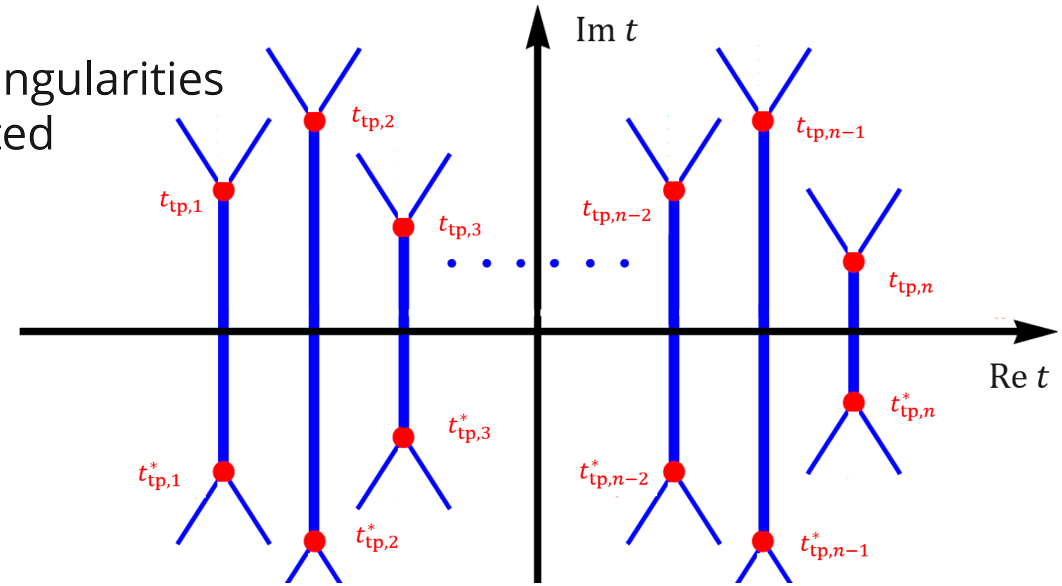

Step 2: Compute Borel sum $\Psi_\pm$ at each Stokes region

- Borel sum is well-defined and computable in each region separated by Stokes lines

$$\Psi_\pm = \int_0^\infty \frac{\mathrm{d}\eta}{\hbar}\, \mathrm{e}^{-\eta/\hbar} B[\psi_\pm](t;\eta) \sim \exp\left[\mp \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \mathrm{d}t' \sqrt{Q(t')}\right] \times (1 + O(\hbar))$$ at each Stokes region

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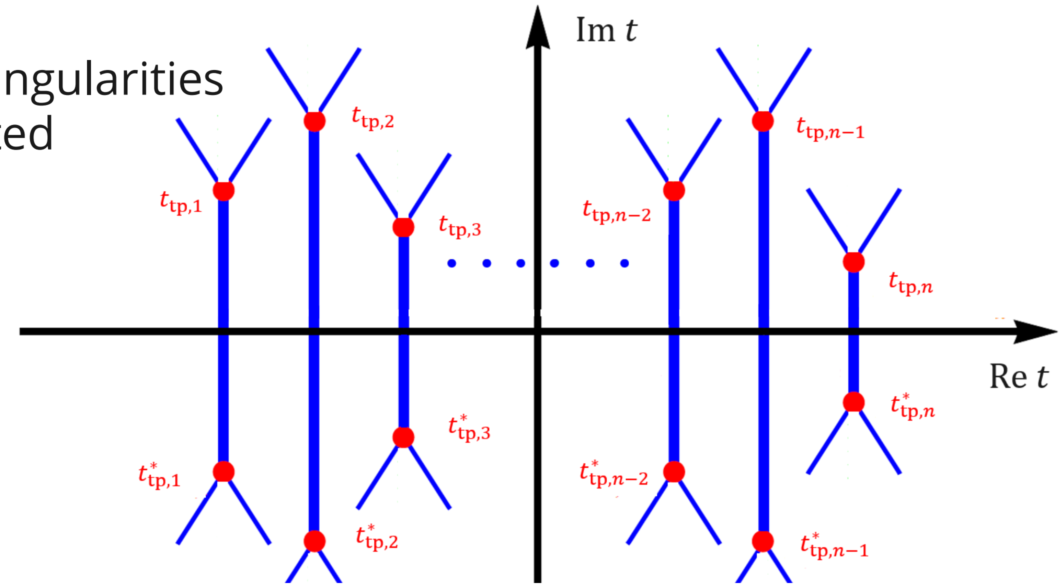

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Step 3: Compute the Stokes constants α and β

- Whenever crosses Stokes lines, $\Psi_{\pm}$ jumps discontinuously (**Stokes phenomenon**)

⇒ The discontinuity is given by the integral of singularities of $B[\psi_{\pm}]$

$$\Psi_{+}(\text{region } A) = \alpha\, \Psi_{+}(\text{region } B) + \beta\, \Psi_{-}(\text{region } B)$$

$$\sim \oint_{\substack{\text{sing. on} \\ \text{a Stokes line}}} \frac{\mathrm{d}\eta}{\hbar}\, \mathrm{e}^{-\eta/\hbar} B[\psi_{\pm}](t;\eta)$$

Contents

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where $t_n^{\mathrm{cr}} = \{t \in \mathbb{R} \mid 0 = \mathrm{Im}\,\mathrm{i}\int_{t_n^{\mathrm{tp}}}^t \omega_{\boldsymbol{p}-e\boldsymbol{A}}\}$ is the crossing b/w n-th Stokes line and the real axis

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Negative plane wave
(0^{th} WKB sol.)

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Negative plane wave (0^{th} WKB sol.)

# of Stokes lines crossed

Stokes const.

Positive plane wave (0^{th} WKB sol.)

⇒ Reproduces the known semi-classical formulas for the particle prod. # at out-state

e.g., Worldline instanton method [Dunne, Schubert, …];
Steepest descent analyses [Izykson-Zuber, Dykene-Davis-Pechuaks]
Imaginary-time method [Popov]; Divergent asymptotic series method [Berry-Dingle]

Strong-field QED wavefunction within EWKB

Apply Exact WKB to Dirac eq.: $0 = [\mathrm{i}\hbar\gamma^0\partial_t - \boldsymbol{\gamma}\cdot(\boldsymbol{p} - e\boldsymbol{A}) - m]\psi(t)$

Step 1: Draw a Stokes graph

Step 2: Compute Borel sum $\Psi_\pm$ at each Stokes region

Step 3: Compute the Stokes constants α and β

At the leading order in $\hbar$

$$\Psi_{-,\boldsymbol{p}}(t) = \left[v_{\boldsymbol{p}-e\boldsymbol{A}} e^{+\mathrm{i}\int^t \omega_{\boldsymbol{p}-e\boldsymbol{A}}} - \sum_n \Theta(t - t_n^{\mathrm{cr}})(-1)^n e^{-\frac{\sigma_n}{\hbar}} u_{\boldsymbol{p}-e\boldsymbol{A}} e^{-\mathrm{i}\int^t \omega_{\boldsymbol{p}-e\boldsymbol{A}}} \right] \left(1 + O(\hbar) + O(e^{-\frac{2}{\hbar}\mathrm{Re}\,\sigma})\right)$$

where $t_n^{\mathrm{cr}} = \{t \in \mathbb{R} \mid 0 = \mathrm{Im}\,\mathrm{i}\int_{t_n^{\mathrm{tp}}}^{t} \omega_{\boldsymbol{p}-e\boldsymbol{A}}\}$ is the crossing b/w n-th Stokes line and the real axis

$\sigma_n = -2\mathrm{i}\int_0^{t_n^{\mathrm{tp}}} \mathrm{d}t\, \omega_{\boldsymbol{p}-e\boldsymbol{A}}$ is the "instanton" action associated to n-th Stokes line

Negative plane wave (0^{th} WKB sol.)

# of Stokes lines crossed

Stokes const.

Positive plane wave (0^{th} WKB sol.)

⇒ Reproduces the known semi-classical formulas for the particle prod. # at out-state

e.g., Worldline instanton method [Dunne, Schubert, …];
Steepest descent analyses [Izykson-Zuber, Dykene-Davis-Pechuaks]
Imaginary-time method [Popov]; Divergent asymptotic series method [Berry-Dingle]

$\Psi_{-,p}$ enables us to discuss various processes in the presence of vacuum particle prod.

HHG: Harmonic spectrum

Setup: A monochromatic E-field $E(t) = E_0 \cos(\Omega t)$

Note that $\omega \frac{\mathrm{d}N}{\mathrm{d}\omega} \sim \left|\omega \tilde{J}(\omega)\right|^2$

What I computed:

VEV of current $\tilde{J}(\omega) = \mathrm{F.T.}[J(t)] = \mathrm{F.T.}\left[\left\langle 0;\mathrm{in}\left|\hat{\bar{\psi}}\gamma^\mu\hat{\psi}\right|0;\mathrm{in}\right\rangle\right]$

using **numerics** and **analytically with the EWKB wavefunc.** $\Psi_{-,p}$ **at LO in** $\hbar$

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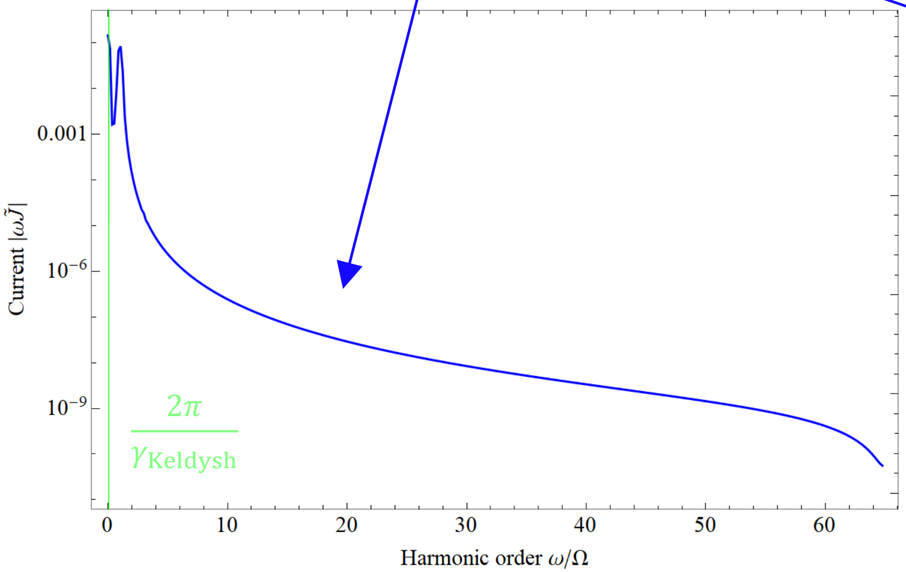
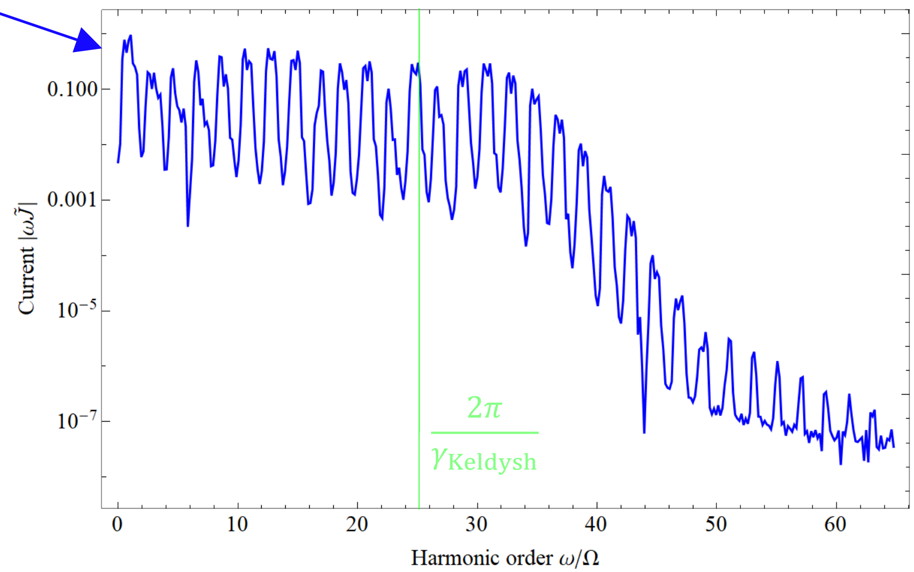
Note that $\omega \frac{\mathrm{d}N}{\mathrm{d}\omega} \sim \left|\omega\tilde{J}(\omega)\right|^2$

weak & fast $\gamma_{\mathrm{Keldysh}} := \frac{m\Omega}{eE_0} = 100$

strong & slow $\gamma_{\mathrm{Keldysh}} = 1/4$





- **Odd high-harmonic peaks appear as decreasing $\gamma_{\mathrm{Keldysh}}$**

  ⇒ Plateau (cutoff) appears when the physics becomes nonperturbative (perturbative)

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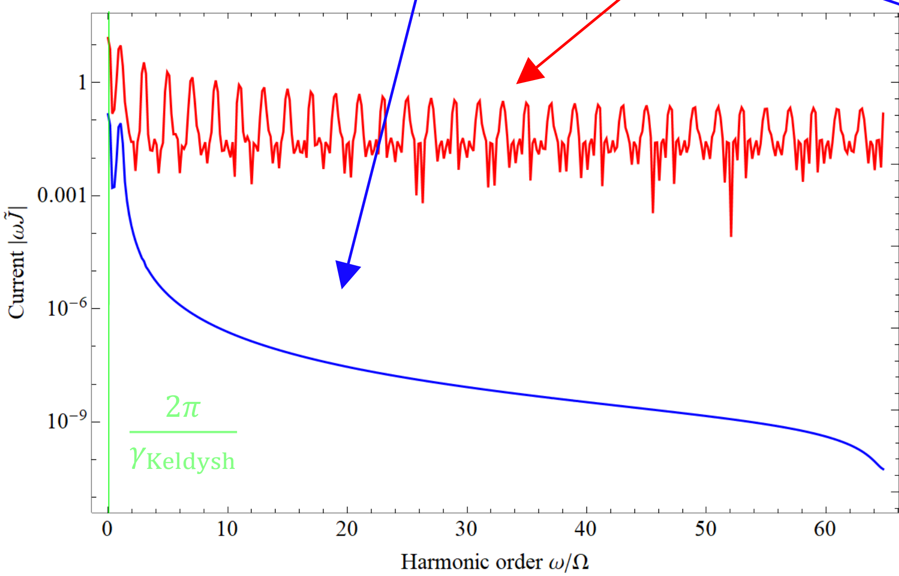
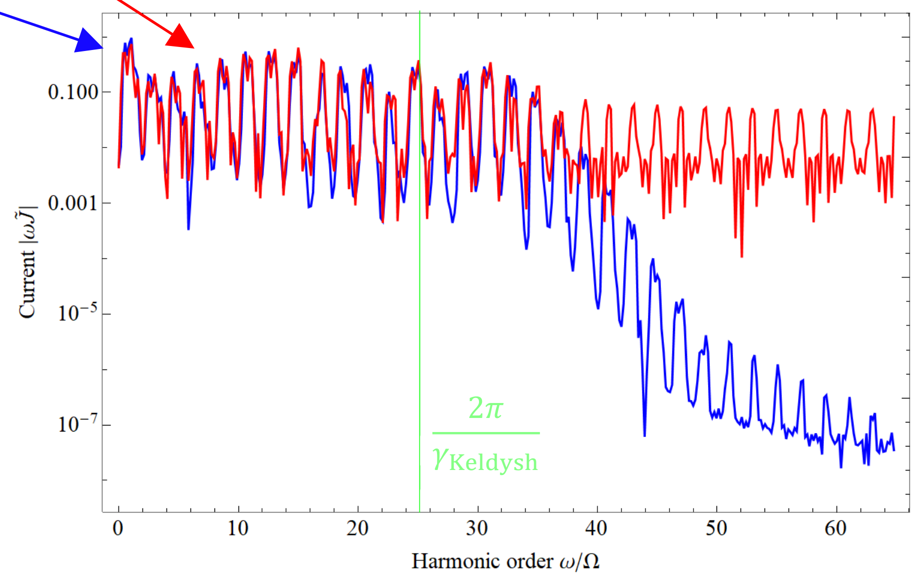




- **Odd high-harmonic peaks appear as decreasing $\gamma_{\mathrm{Keldysh}}$**

  ⇒ Plateau (cutoff) appears when the physics becomes nonperturbative (perturbative)

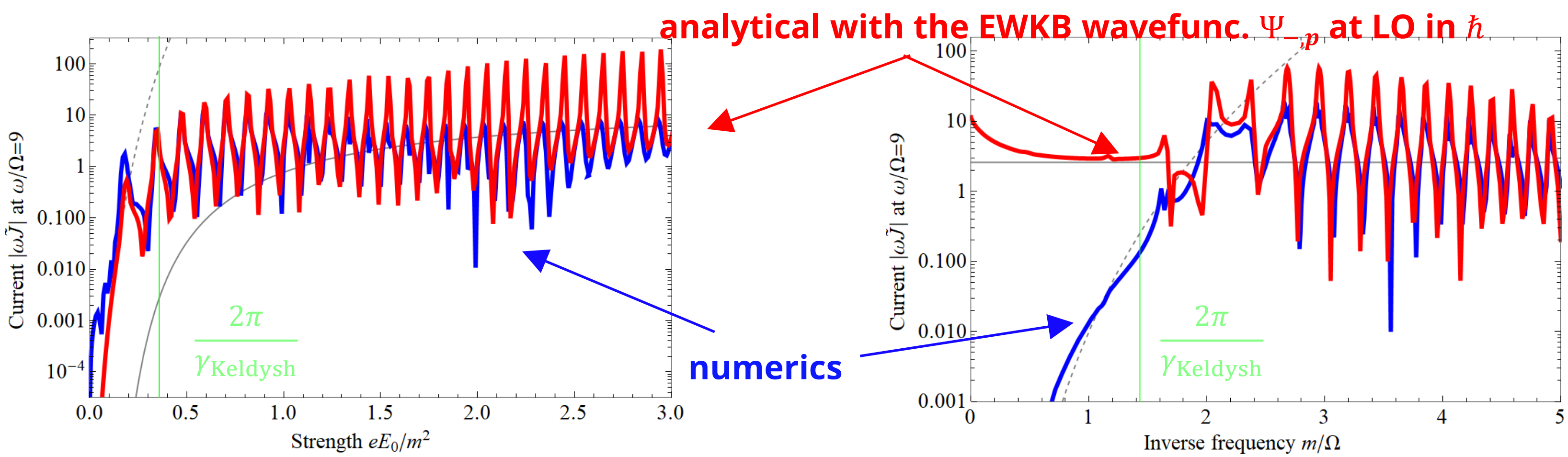
- **LO Exact WKB is very GOOD for the plateau, but is BAD after the cutoff**

  ⇒ LO approx. is GOOD for nonpert. processes but BAD for pert. processes (e.g., multi-photon)

HHG: Harmonic intensity

✔ **Magnitude of the harmonic peak at $\omega/\Omega = 9$**

Dependence on strength E_0	Dependence on frequency Ω



- **In the nonpert. regime (strong E_0 & small Ω), the harmonic intensity:**
  - saturates ⇒ resulting in the plateau
  - oscillates ⇒ consistent with recent semi-conductor exp. [Xia *et al.*, (2020)]

    (but only E_0-dep. is measured and Ω-dep. is our prediction)
- **The <u>saturation</u> & <u>oscillation</u> are nicely captured by LO Exact WKB**

Exact WKB is a nice analytical approach for nonpertubative processes in the presence of vacuum particle production

Contents

Summary

Problem: High-harmonic generation (HHG) from the vacuum by strong time-dependent & spatially-homogeneous electric field

Approach: Exact WKB method

✔ **Exact WKB is a nice analytical method to construct wavefunc. under strong fields, including Stokes phenomenon** (i.e., dynamics of vacuum particle prod.)

✔ **Particle production from the vacuum leads to high-harmonic generation:**

- Only odd harmonics are generated
- Plateau structure, whose cutoff is set by the Keldysh parameter $\gamma_{\rm Keldysh} = \frac{m\Omega}{eE_0}$
- Saturation and oscillation in the nonperturbative regime $\gamma_{\rm Keldysh} \lesssim 1$

✔ The nonperturbative features of HHG are nicely reproduced by Exact WKB even with LO treatment in $\hbar$

⇒ **Exact WKB (or WKB) is a powerful tool to describe strong-field QED phenomena !**

cf. Antonino's talk (w/o Stokes pheno. but w/ spatial inhomo.)

For more details:

[1] HT, Fujimori, Misumi, Nitta, Sakai, JHEP 03, 082 (2021) [2010.16080]

[2] HT, Hongo, Ikeda, 2105.12446

(Note that [2] is written for cond-mat.,
but the basic idea/calculation is the same for strong-field QED)

Intuitive picture

(a)

(b)

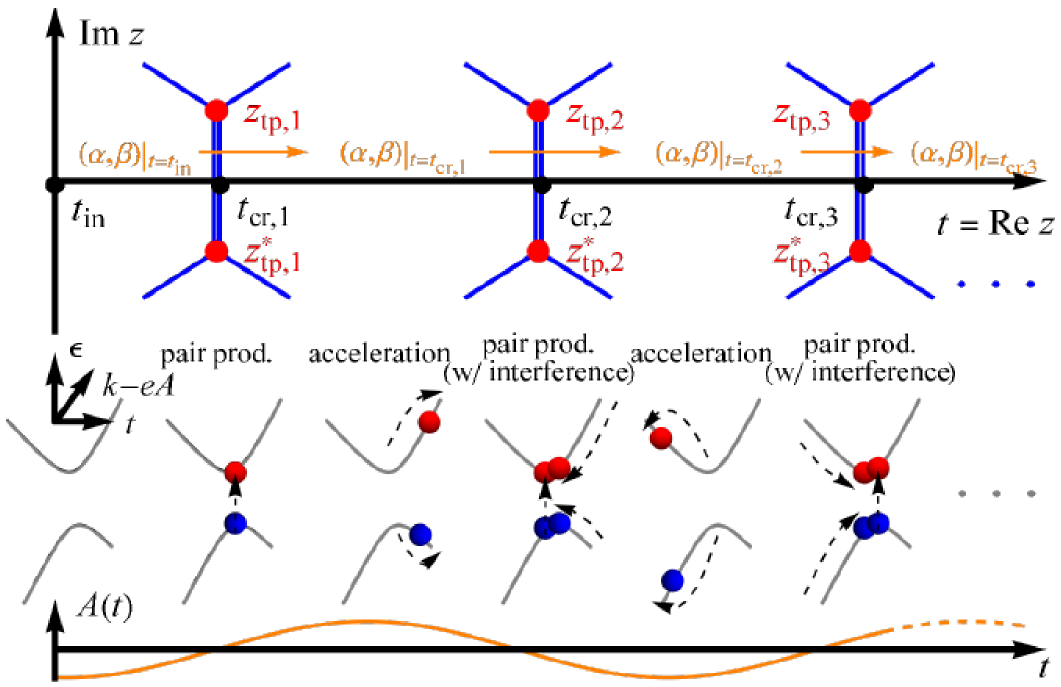
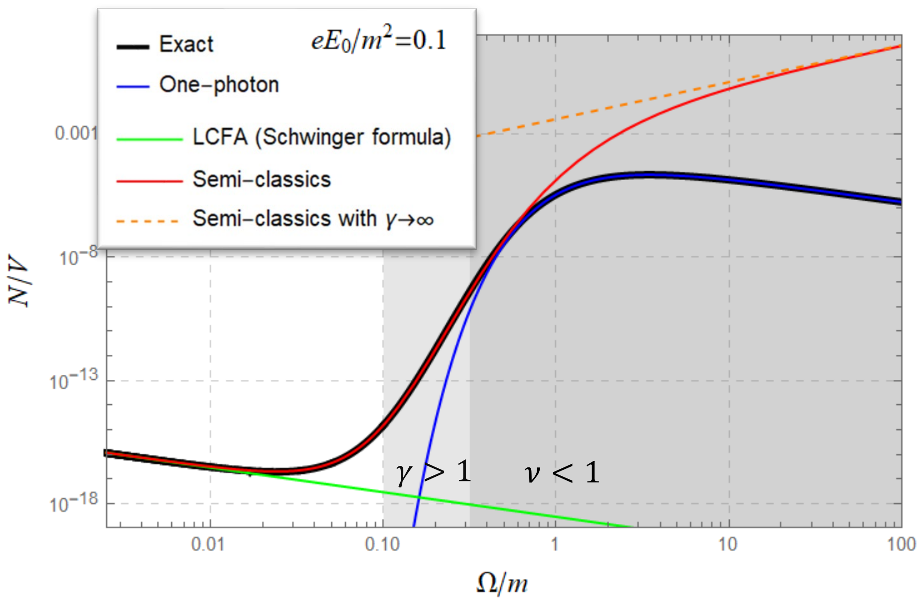


FIG. 1. (a) A typical Stokes graph, composed of Stokes lines (blue lines) and turning points (red points), and (b) the corresponding physical processes during the real-time evolution.

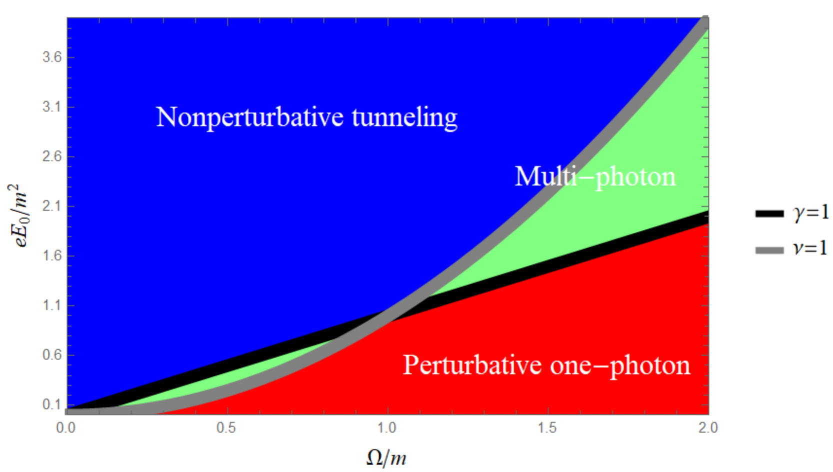
Interplay b/w tunneling, multi-, one-photon

✔ **An explicit demonstration for Sauter pulsed E-field** $eE(t) = \frac{eE_0}{\cosh^2(\Omega t)}$



- **One-photon dominates for large Ω**, where semi-classical methods fail,
- **One-photon is more efficient** than tunneling ; N is the largest at $\Omega \sim 2m$

✔ **Two dimensionless parameters** $\gamma = \frac{m\Omega}{eE_0}$, $\nu \equiv \frac{eE_0}{\Omega^2}$ **control the interplay**



- Sauter field is solvable, which can be compared w/ Schwinger & one-photon

$$\gamma \gg 1,\ \ \nu \ll 1 \Rightarrow \text{ pert. one−photon}$$
$$\gamma \ll 1,\ \ \nu \gg 1 \Rightarrow \text{non−pert. tunneling}$$

- $\nu = \frac{eE_0/\Omega}{\Omega} = \frac{\text{(work done by E-field)}}{\text{(photon energy)}}$

$= \text{(\# of photons involved)}$

Intra- and inter-band contributions

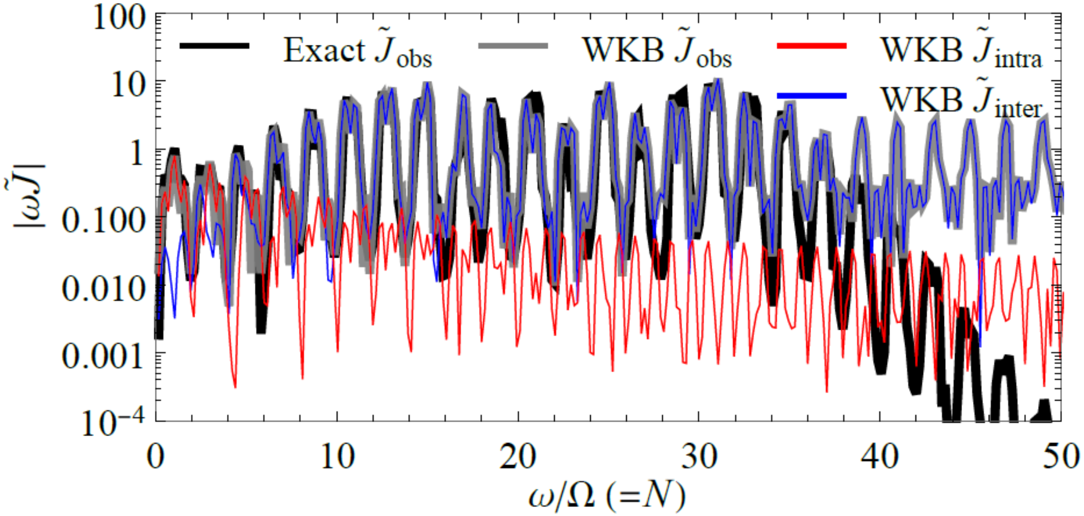


FIG. 2. HHG spectrum for the oscillating field (10), with $\Omega/(\Delta/2) = 1/4$, $eE_0/(\Delta/2)^2 = 1$ (i.e., $\gamma = 1/4$) and $\Omega t_{\rm in} = -17\pi/3$, $T_w = 2|t_{\rm in}|$. The parameter set corresponds to, e.g., $\Omega/2\pi = 1\,{\rm THz}$ and $E_0 = 4.2\,{\rm kV/cm}$ for a Dirac material with Fermi velocity $v_{\rm F} = 10^6\,{\rm m/s}$ and mass $\Delta = 33\,{\rm meV}$.

Analytical expression for the current

$$\tilde{J}_{\text{intra}} \sim \mathrm{e}^{-\frac{2\,\mathrm{Re}\,\sigma}{\hbar}} \frac{-\mathrm{i}}{\pi \cos^2 \frac{\theta}{2}} \sum_{n=-\infty}^{\infty} \left[\sum_{\pm} \mathrm{e}^{+\mathrm{i}\pi\left(\left\lceil \frac{t_{\text{in}}}{\pi/\Omega} \right\rceil - \frac{1}{2}\right)\left(\mp \frac{\theta}{\pi} - 1\right)} \frac{\sin \frac{\theta}{2}}{4\left(n - \frac{\theta}{2\pi}\right)} \tilde{W}\left(\omega - \Omega \mp 2\left(n - \frac{\theta}{2\pi}\right)\Omega\right) - \frac{\tilde{W}\left(\omega - (2n-1)\Omega\right)}{2n-1} \right],$$

$$\tilde{J}_{\text{inter}} \sim \mathrm{e}^{-\frac{\mathrm{Re}\,\sigma}{\hbar}} \frac{-\mathrm{i}(-1)^{\left\lceil \frac{t_{\text{in}}}{\pi/\Omega} \right\rceil}\gamma}{4\pi \cos \frac{\theta}{2}} \sum_{n=-\infty}^{+\infty} \sum_{\pm} \left[\left(\ln \gamma^2 + 2H_{\pm n - 1/2}\right) \mathrm{e}^{+\mathrm{i}\pi\left(\left\lceil \frac{t_{\text{in}}}{\pi/\Omega} \right\rceil - \frac{1}{2}\right)\left(\mp \frac{\theta}{\pi} - 1\right)} \tilde{W}\left(\omega \mp 2\left(n - \frac{\theta}{2\pi}\right)\Omega\right) \right]$$

$$+ \left(\ln\left(4\gamma^2\right) - \left((-1)^n \cos\frac{\theta}{2} - 1\right) H_{\frac{n}{2} \pm \frac{\theta}{4\pi} - 1} + \left((-1)^n \cos\frac{\theta}{2} + 1\right) H_{\frac{n}{2} \pm \frac{\theta}{4\pi} - \frac{1}{2}} \right) \tilde{W}\left(\omega - (2n-1)\Omega\right) \Big], \qquad (11)$$