Schwinger mechanism with time-dependent electric fields

Hidetoshi TAYA

RIKEN iTHEMS

- Interplay between perturbative & non-perturbative particle production:
  [HT, Fujii, Itakura, PRD (2014)] [HT, Fujimori, Misumi, Nitta, Sakai, JHEP (2021)]
- Dynamically assisted Schwinger mechanism and Furry-picture perturbation theory:
  [HT, PRD (2019)] [Huang, HT, PRD (2019)]
- Review on strong-field QED:
  [Fedotov, Ilderton, Karbstein, King, Seipt, HT, Torgrimsson, 2203.00019]

What is the Schwinger mechanism?

[Sauter (1932)] [Heisenberg, Euler (1936)] [Schwinger (1951)]

✔ **Vacuum pair production occurs in the presence of strong E-fields**

Intuitive picture for a slow electric field = **quantum tunneling**

What is the Schwinger mechanism?

[Sauter (1932)] [Heisenberg, Euler (1936)] [Schwinger (1951)]

✔ Vacuum pair production occurs in the presence of strong E-fields

Intuitive picture for a slow electric field = **quantum tunneling**

✔ For constant strong E-field $E(t, x) = E$, it's understood well (at least, theoretically)

$$N_{e^\pm} = \frac{(eE)^2 VT}{(2\pi)^3} \times \exp\left[-\pi \frac{m^2}{eE}\right] \sim \exp[-\# \times (\text{gap height}) \times (\text{gap length})]$$

[Schwinger (1951)]

- Non-perturbative ⇒ Interesting, since it is the unexplored region of QED (or QFT)
- Vacuum process ⇒ Fundamental, since all the physical processes occur on top of vacuum
- However, exponentially suppressed i.t.o. mass ⇒ Not confirmed by experiments yet…

cf.) Guinness world record: [Yanovsky et al (2008)]

HERCULES laser $eE \sim \sqrt{10^{22}\ \mathrm{W/cm^2}} \sim (0.01\ \mathrm{MeV})^2 \ll m_e^2 \sim \sqrt{10^{29}\ \mathrm{W/cm^2}} \sim (0.5\ \mathrm{MeV})^2$

Timeliness

✔ **Now is the best time to study the Schwinger mechanism !**

Developments in experimental technologies ⇒ **Novel strong-field sources**

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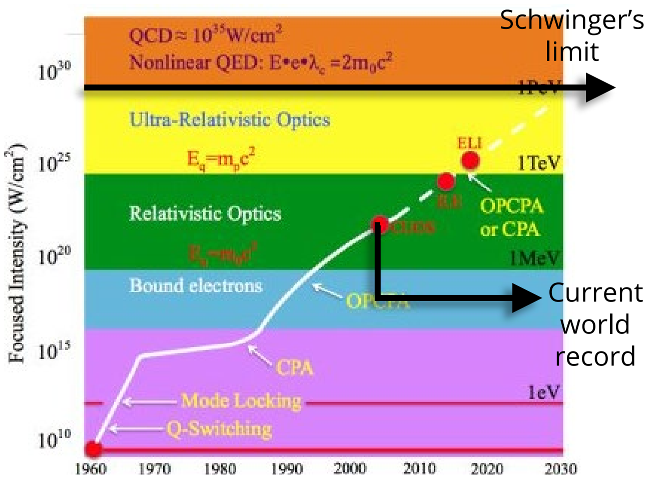

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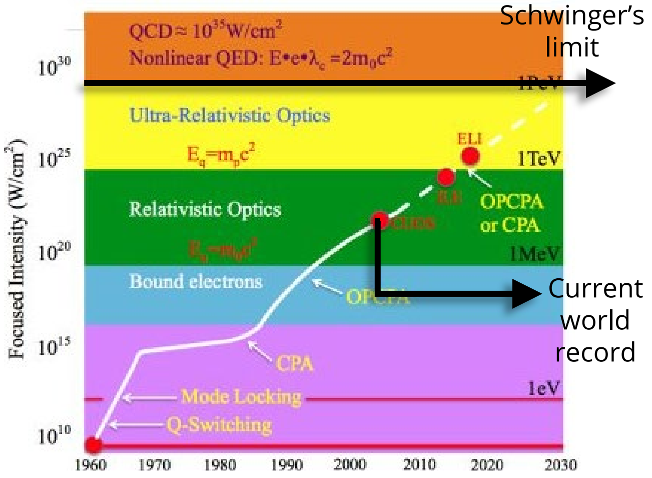

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ex.2) Heavy-ion collisions

- (ultra)peripheral collisions

$$eE, eB \sim \frac{\alpha Z v \gamma}{r^2} \sim \alpha Z \gamma \times m_\pi^2$$

[Skokov, Illarionov, Toneev (2009)]
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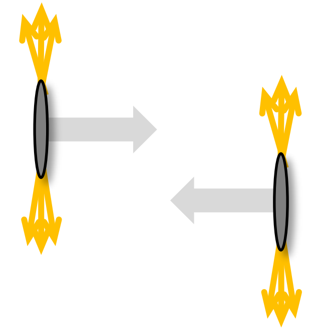
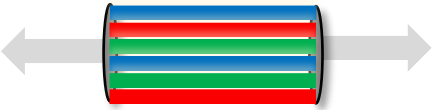


- glasma (color flux tube)

$$gE_{\rm color}, gB_{\rm color} \sim Q_s^2 \sim O(1)\ {\rm GeV}^2$$



[Lappi, McLerran (2006)]

- Also in other geometries:
  - asymmetric coll. [Hirono, Hongo, Hirano (2014)] [Voronyuk, et al (2014)]
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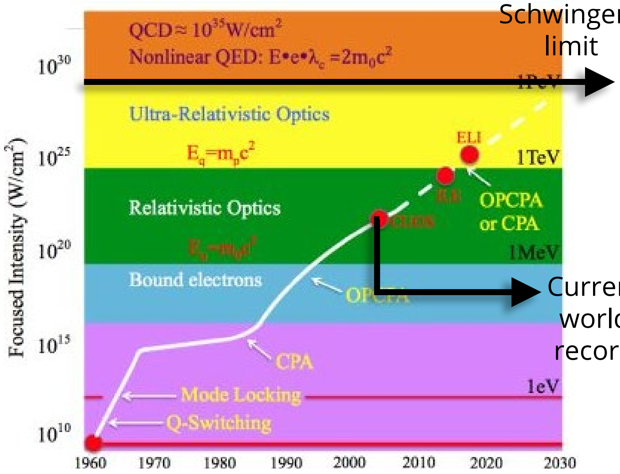

Schwinger's limit

Current world record

[Mourou, Tajima, Bulanov (2006)]

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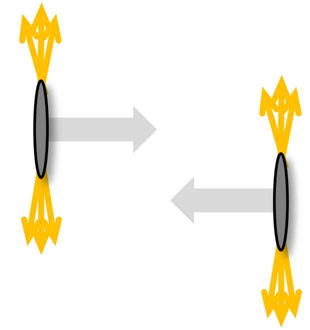

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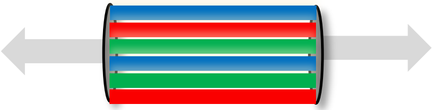

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[Allor, Cohen, McGady (2008)] [Solinas, Amoretti, Giazotto (2021)]

ex.3) Cond-mat analogues: Graphene, Cold atom, Superconductor, Semiconductor, …

[Szpak, Shutzhold (2012)] [Thesis by Linder; 1807.08050]

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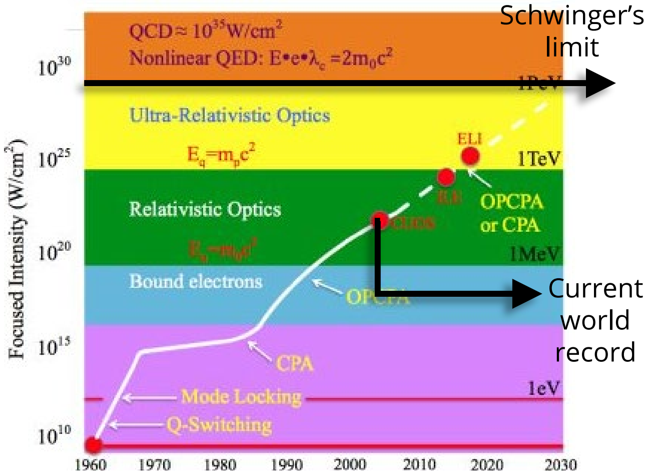

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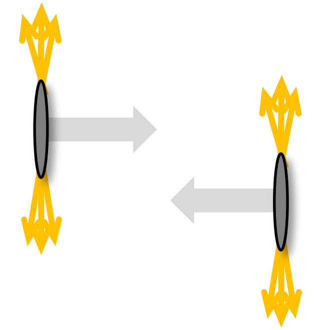

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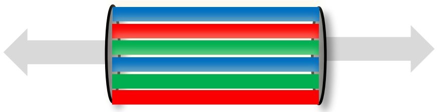

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✔ Schwinger's result is insufficient for actual problems ⇒ **Beyond Schwinger !**

- Inhomogeneous field (e.g., time- & space-dependence, more efficient field config., …)
- Realtime dynamics (e.g., backreaction, intermediate particle number, …)
- Higher-loop effects (e.g., radiation, mass shift, exponentiation conjecture, …)
- Other observables (e.g., spin, chirality, high-harmonic generation, …)
- …

Today's talk

Topic: the Schwinger mechanism with time-depending E-field

Part I: Interplay b/w non-pert. & pert. production mechanisms

[HT, Fujii, Itakura, PRD (2014)] [HT, Fujimori, Misumi, Nitta, Sakai, JHEP (2021)]

Part II: Dynamically assisted Schwinger mechanism

[HT, PRD (2019)] [Huang, HT, PRD (2019)]

The most important message

Perturbation theory in the Furry picture gives you a very powerful analytical approach for the Schwinger mechanism with time-dependent E-field

Introduction

⇓

Part I:

Interplay b/w non-pert. & pert. production mechanisms

⇓

Part II:

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⇓

Summary

Background (1/2): Interplay b/w non-pert. & pert. mech.

✓ Consider time-dependent E-field, having strength eE_0 and frequency Ω

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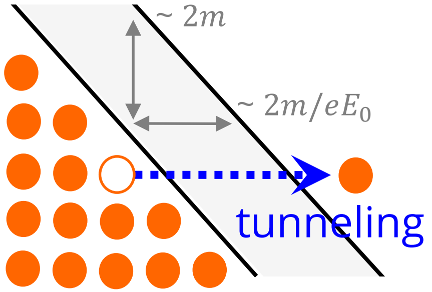
✔ **Consider time-dependent E-field, having strength eE_0 and frequency Ω**

- The tunneling picture should break down if the frequency Ω is large

Non-pert. tunneling $N \sim \exp[\#/eE_0]$



Tunneling time $\Delta t \sim \frac{2m}{eE_0}$

$\Rightarrow$ E-field should be slower than Δt

$\Rightarrow \Omega^{-1} \gtrsim \Delta t$

$\Rightarrow 1 \gtrsim \frac{\Delta t}{\Omega^{-1}} = \frac{\Omega m}{eE_0} \equiv \gamma$ (Keldysh parameter)

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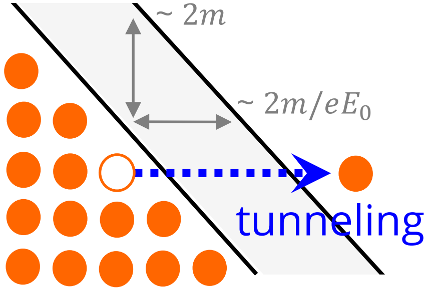

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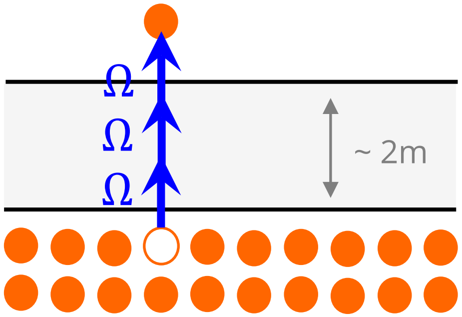
Pert. multi-photon $N \sim eE_0^{2n}$



Pair production when $n\Omega > 2m$

- For large Ω, E-field may behave like a photon and interact incoherently (perturbatively)
  ⇒ an analogue of photo-electric effect in materials

Background (2/2): Semi-classical analysis

✔ The interplay is said to be “confirmed” by (but only by) semi-classical analysis

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- Trans-series expansion in terms of $\hbar$

$$N = \sum_{n,m} N_{n,m} \hbar^n \mathrm{e}^{-mS/\hbar} = \left(N_{0,1} + O(\hbar)\right) \mathbf{e}^{-S/\hbar} + O(\mathrm{e}^{-2S/\hbar})$$

- Steepest descent analysis [Brezin, Itzykson (1970)]
- Imaginary-time method [Popov (1972)]
- Divergent asymptotic series method [Berry (1989)]
- Worldline instanton method [Dunne, Shubert (2005)]
- (exact) WKB [HT, Fujimori, Misumi, Nitta, Sakai (2020)]
- ...

- Valid in the slow limit $\hbar \ll 1 \Leftrightarrow \Omega \ll 1$: $\because \ \mathrm{i}\hbar\partial_t\psi = H(\Omega t)\psi \xrightarrow[\tau \equiv t/\hbar]{} \mathrm{i}\partial_\tau\psi = H(\hbar\Omega\tau)\psi$

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- Production number N is controlled solely by the Keldysh parameter $\gamma \equiv \frac{m\Omega}{eE_0}$ [Keldysh (1965)]

$$S = \pi\frac{m^2}{eE_0}g(\gamma) \xrightarrow[\text{Example: } eE(t) = eE_0\cos(\Omega t)]{} \begin{cases} \pi\frac{m^2}{eE_0} + O(\gamma^1) \Rightarrow N \propto \mathrm{e}^{-\pi\frac{m^2}{eE_0}} & (\text{slow limit } \gamma \ll 1) \\ \frac{2m}{\Omega}\ln\gamma^2 + \text{const.} + O(\gamma^{-1}) \Rightarrow N \propto \left(\frac{eE_0}{m^2}\right)^{\frac{2m}{\Omega}} & (\text{fast limit } \gamma \gg 1) \end{cases}$$

See, e.g., [Dunne, Gies, Schubert, Wang (2006)]
for an explicit expression for $g(\gamma)$

# of photons

pert. power-dep.
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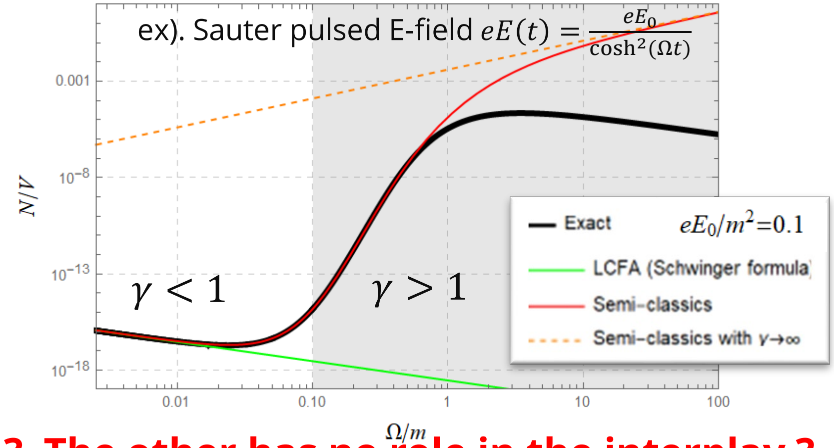
✔ However, you should not be convinced with the semi-classical argument

- Fast limit $\gamma \gg 1$ is quite dangerous

⇒ Q1: What is really happening at $\gamma \gg 1$?



- Another dimensionless parameter should exist

$\because$ 3 dimension**ful** parameters m, eE_0, Ω

⇒ 2 dimension**less** parameters **⇒ Q2: Why only γ ? The other has no role in the interplay ?**

Idea: One-photon process

- ✓ **One-photon process is the key to answer Q1 & Q2 !**

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✔ One-photon process is the key to answer Q1 & Q2 !

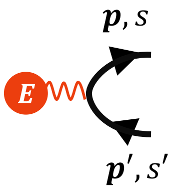
✔ Larger Ω, more perturbative ⇒ one-photon dominates eventually

- because there is no time to interact many times, no matter how strong eE_0 is

multiple scatterings

finite # of scatterings

- cover different region, as # of photons should be huge $\frac{2m}{\Omega} \to \infty$ in the semi-classics
- analytical formula is available (for any field config. !)

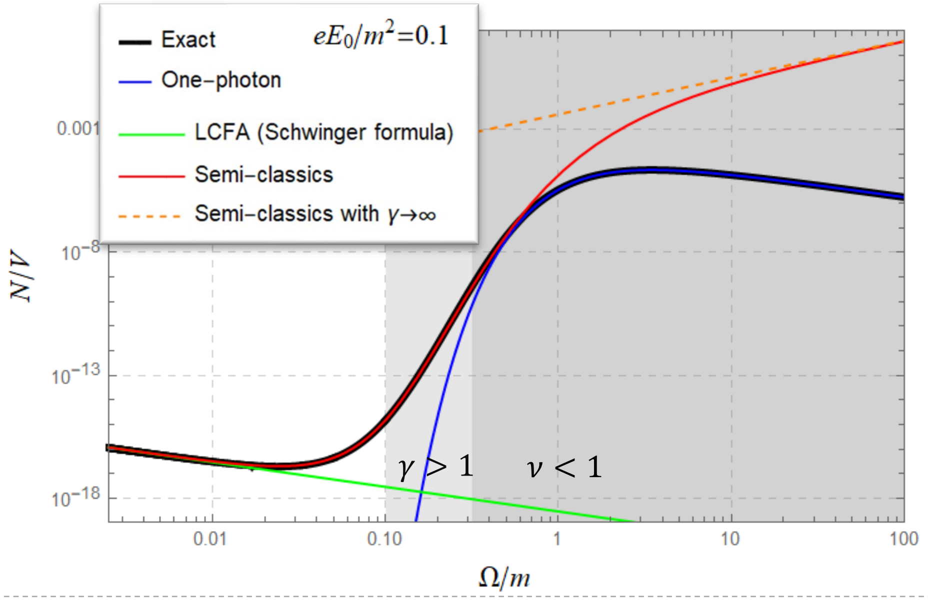


$$N_{1-\text{photon}} = \sum_{s,s'} \int \mathrm{d}^3\boldsymbol{p}\, \mathrm{d}^3\boldsymbol{p}' \left| \qquad \right|^2 = \frac{V}{(4\pi)^2} \int_{2m}^{\infty} \mathrm{d}\omega \sqrt{1 - \frac{4m^2}{\omega^2}} \frac{1}{3} \left(2 + \frac{4m^2}{\omega^2}\right) \left|e\tilde{E}(\omega)\right|^2$$

[HT, Fujii, Itakura (2014)] [QFT textbook by Itzykson, Zuber]

Result: Interplay b/w tunneling, multi-, one-photon

✔ **An explicit demonstration for Sauter pulsed E-field** $eE(t) = \frac{eE_0}{\cosh^2(\Omega t)}$



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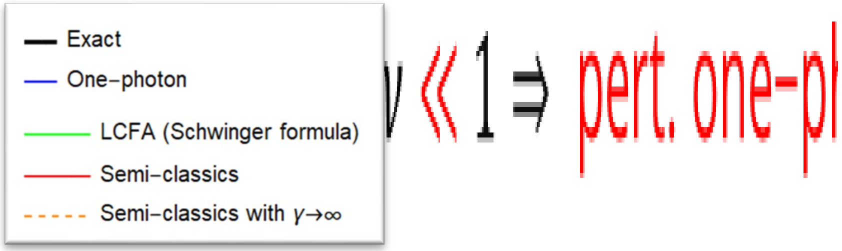

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$\gamma > 1$ $\nu < 1$

✔ **Two dimensionless parameters $\gamma = \frac{m\Omega}{eE_0}$, $\nu \equiv \frac{eE_0}{\Omega^2}$ control the interplay**

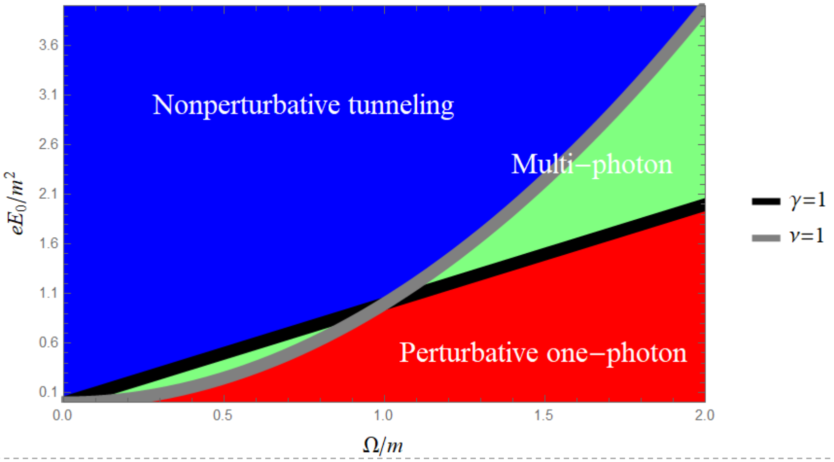
- Sauter field is solvable, which can be compared w/ Schwinger & one-photon

$\gamma \gg 1,\ \nu \ll 1 \Rightarrow$ pert. one−photon

$\gamma \ll 1,\ \nu \gg 1 \Rightarrow$ non−pert. tunneling

- $\nu = \frac{eE_0/\Omega}{\Omega} = \frac{\text{(work done by E-field)}}{\text{(photon energy)}}$

  $=$ (# of photons involved)



Message of Part I

(1) The interplay b/w non-pert. & pert. production mechanisms is controlled

by $\gamma = \frac{m\Omega}{eE}$ (Keldysh parameter) and also by $\nu = \frac{eE}{\Omega^2}$ (~ # of photons involved)

(2) Semi-classical methods (e.g., worldline) are dangerous for $\gamma \gg 1$;

It breaks down for $\nu \gtrsim 1$, where one-photon process dominates

(3) One-photon pert. production is very efficient, compared to non-pert. tunneling

⇒ This is the essence of the dynamically assisted Schwinger mech., as I show from now

[HT, Fujii, Itakura, PRD (2014)]
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- ✔ **In Part I, I discussed E-field having a single frequency mode Ω**
- ✔ **What if E-field is bi-frequent; superposition of slow strong + weak fast E-fields ?**

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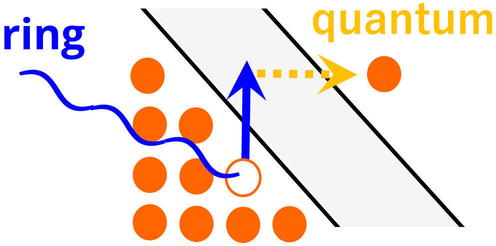
- Combinatorial effects of non-pert. & pert. mechs. ⇒ **dynamically assisted Schwinger mech.**



perturbative scattering

quantum tunneling

[Dunne, Gies, Schutzhold (2008), (2009)]
[Piazza et al (2009)] [Monin, Voloshin (2010)]

$N \sim \exp[-\# \times (\text{gap height}) \times (\text{gap length})]$ ⇒ **Enhancement in production**
(even though the fast field is very weak)

reduced by pert.

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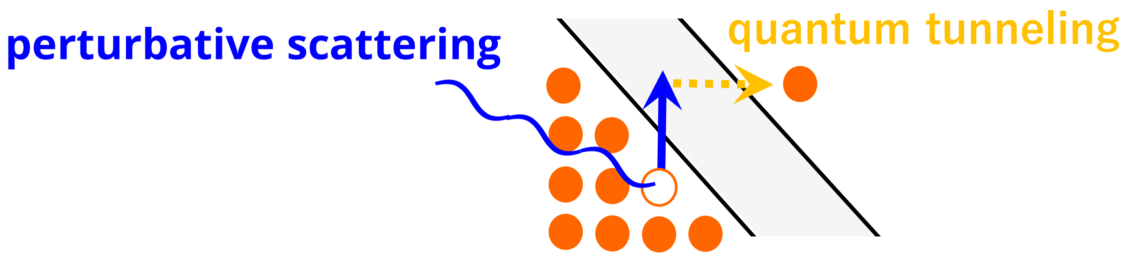

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ex1) laser: available E-field is still weak ⇒ needs enhancement to observe the Schwinger mech

ex2) heavy-ion collisions: (Mini-)jets on top of glasma, Event generators (e.g., PYTHIA)

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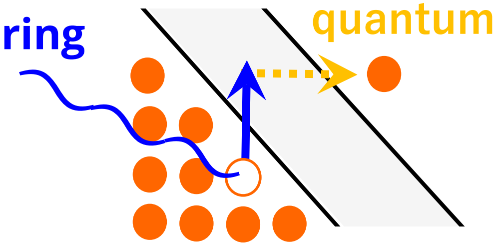

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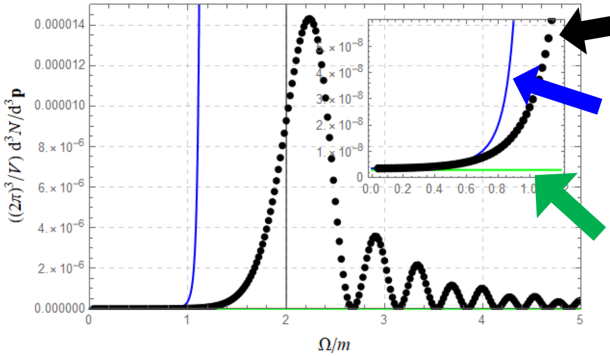
- Typically, analyzed within semi-classical methods

BUT: NOT quantitatively good

Q: Better method ???

⇒ A: Perturbation theory in the Furry picture



Exact numerical result

Semi-classical result

Schwinger result for const. E-field

Perturbation theory in Furry picture

Idea

Split the field E into strong E_S and weak E_w parts and treat E_S and E_w non-perturbatively and perturbatively, respectively.

cf. distorted Born wave approximation in quantum mechanics

Perturbation theory in Furry picture (1/3)

STEP 1 **Split the total $\boldsymbol{E}$ into strong slow $\boldsymbol{E}_\mathrm{s}$ & weak fast $\boldsymbol{\mathcal{E}}_\mathrm{w}$**

$$\boldsymbol{E} = \boldsymbol{E}_\mathrm{s} + \boldsymbol{\mathcal{E}}_\mathrm{w}$$

Perturbation theory in Furry picture (1/3)

STEP 1

Split the total $\boldsymbol{E}$ into strong slow $\boldsymbol{E}_{\mathrm{s}}$ & weak fast $\boldsymbol{\mathcal{E}}_{\mathrm{w}}$

$$\boldsymbol{E} = \boldsymbol{E}_{\mathrm{s}} + \boldsymbol{\mathcal{E}}_{\mathrm{w}}$$

STEP 2

Solve Dirac eq. non-pert. w.r.t. $\boldsymbol{E}_{\mathrm{s}}$, but just pert. w.r.t. $\boldsymbol{\mathcal{E}}_{\mathrm{w}}$

$$[i\not{\partial} - \boldsymbol{e}\not{\boldsymbol{A}}_{\mathrm{s}} - m]\hat{\psi} = \boldsymbol{e}\not{\boldsymbol{\mathcal{A}}}_{\mathrm{w}}\hat{\psi}$$

$$\Rightarrow \hat{\psi}(x) = \hat{\psi}^{(0)}(x) + \int_{-\infty}^{\infty} \mathrm{d}y^4 S_{\mathrm{R}}(x,y)\boldsymbol{e}\not{\boldsymbol{\mathcal{A}}}_{\mathrm{w}}(y)\hat{\psi}^{(0)}(y) + O(|\boldsymbol{e}\boldsymbol{\mathcal{A}}_{\mathrm{w}}|^2)$$

Here, $\hat{\psi}^{(0)}$ and S_{R} are **non-perturbatively dressed by $\boldsymbol{E}_{\mathrm{s}}$** as

$$[i\not{\partial} - \boldsymbol{e}\not{\boldsymbol{A}}_{\mathrm{s}} - m]\hat{\psi}^{(0)} = 0$$

$$[i\not{\partial} - \boldsymbol{e}\not{\boldsymbol{A}}_{\mathrm{s}} - m]S_{\mathrm{R}}(x,y) = \delta^4(x-y)$$

Perturbation theory in Furry picture (2/3)

STEP 3

Compute in/out annihilation operators $\hat{a}^{\mathrm{in/out}}_{p,s}$, $\hat{b}^{\mathrm{in/out}}_{p,s}$ from $\hat{\psi}$

$$\begin{pmatrix} \hat{a}^{\mathrm{in/out}}_{\boldsymbol{p},s} \\ \hat{b}^{\mathrm{in/out}\dagger}_{-\boldsymbol{p},s} \end{pmatrix} \equiv \lim_{t\to-\infty/+\infty} \int \mathrm{d}^3\boldsymbol{x} \begin{pmatrix} (u_{\boldsymbol{p},s}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}})^\dagger \\ (v_{\boldsymbol{p},s}\mathrm{e}^{+i\omega_{\boldsymbol{p}}t}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}})^\dagger \end{pmatrix} \hat{\psi}(x)$$

Perturbation theory in Furry picture (2/3)

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⇒ $\hat{o}^{\text{in}}_{\boldsymbol{p},s}$, $\hat{o}^{\text{out}}_{\boldsymbol{p},s}$ **are inequivalent** $\hat{o}^{\text{in}}_{\boldsymbol{p},s} \neq \hat{o}^{\text{out}}_{\boldsymbol{p},s}$ and related with each other by a Bogoliubov transformation

$$\begin{pmatrix} \hat{a}^{\text{out}}_{\boldsymbol{p},s} \\ \hat{b}^{\text{out}\dagger}_{-\boldsymbol{p},s} \end{pmatrix} = \sum_{s'} \int \mathrm{d}^3\boldsymbol{p}' \begin{pmatrix} \alpha_{\boldsymbol{p},s;\boldsymbol{p}',s'} & \beta_{\boldsymbol{p},s;\boldsymbol{p}',s'} \\ -\beta^*_{\boldsymbol{p},s;\boldsymbol{p}',s'} & \alpha^*_{\boldsymbol{p},s;\boldsymbol{p}',s'} \end{pmatrix} \begin{pmatrix} \hat{a}^{\text{in}}_{\boldsymbol{p}',s'} \\ \hat{b}^{\text{in}\dagger}_{-\boldsymbol{p}',s'} \end{pmatrix}$$

where, up to 1st order in $e\mathcal{A}_{\mathrm{w}}$,

$$\alpha_{\boldsymbol{p},s;\boldsymbol{p}',s'} = \int \mathrm{d}^3\boldsymbol{x}\ {}_{+}\psi^{(0)\text{out}\dagger}_{\boldsymbol{p},s}\ {}_{+}\psi^{(0)\text{in}}_{\boldsymbol{p}',s'} - i\int d^4x\ {}_{+}\bar{\psi}^{(0)\text{out}}_{\boldsymbol{p},s} e\mathcal{A}\!\!\!/_{\mathrm{w}}\ {}_{+}\psi^{(0)\text{in}}_{\boldsymbol{p}',s'} + O(|e\mathcal{A}_{\mathrm{w}}|^2)$$

$$\beta_{\boldsymbol{p},s;\boldsymbol{p}',s'} = \int \mathrm{d}^3\boldsymbol{x}\ {}_{-}\psi^{(0)\text{out}\dagger}_{\boldsymbol{p},s}\ {}_{+}\psi^{(0)\text{in}}_{\boldsymbol{p}',s'} - i\int d^4x\ {}_{-}\bar{\psi}^{(0)\text{out}}_{\boldsymbol{p},s} e\mathcal{A}\!\!\!/_{\mathrm{w}}\ {}_{+}\psi^{(0)\text{in}}_{\boldsymbol{p}',s'} + O(|e\mathcal{A}_{\mathrm{w}}|^2)$$

Here, ${}_{\pm}\psi^{(0)\text{in/out}}_{\boldsymbol{p},s}$ are sol. of Dirac eq. **dressed by $e\boldsymbol{A}_{\mathrm{s}}$** w/ different B.C.

$$[i\partial\!\!\!/ - eA\!\!\!/_{\mathrm{s}} - m]\ {}_{\pm}\psi^{(0)\text{in/out}}_{\boldsymbol{p},s} = 0 \quad \text{w/} \lim_{t\to-\infty/+\infty} \begin{pmatrix} {}_{+}\psi^{(0)\text{in/out}}_{\boldsymbol{p},s} \\ {}_{-}\psi^{(0)\text{in/out}}_{\boldsymbol{p},s} \end{pmatrix} = \begin{pmatrix} u_{\boldsymbol{p},s}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \\ v_{\boldsymbol{p},s}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} \end{pmatrix}$$

Perturbation theory in Furry picture (3/3)

STEP 4 Evaluate the in-vacuum expectation value of # operator

$$\frac{\mathrm{d}^3 N_e}{\mathrm{d}\boldsymbol{p}^3} \equiv \langle \mathrm{vac};\mathrm{in} | a^{\mathrm{out}\dagger}_{\boldsymbol{p},s} a^{\mathrm{out}}_{\boldsymbol{p},s} | \mathrm{vac};\mathrm{in} \rangle = \sum_{s'} \int \mathrm{d}^3\boldsymbol{p}' \left| \beta_{\boldsymbol{p},s;\boldsymbol{p}',s'} \right|^2$$

Perturbation theory in Furry picture (3/3)

STEP 4 **Evaluate the in-vacuum expectation value of # operator**

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Assume $\boldsymbol{E}_{\mathbf{s}}$ is so slow that it can be approximated as a static E-field

$\Rightarrow$ analytical sol. of Dirac eq. ${}_{\pm}\psi^{(0)\mathrm{in/out}}_{\boldsymbol{p},s}$ is known

$\Rightarrow$ one can evaluate $\beta_{\boldsymbol{p},s;\boldsymbol{p}',s'}$ **exactly !**

Perturbation theory in Furry picture (3/3)

STEP 4 **Evaluate the in-vacuum expectation value of # operator**

$$\frac{\mathrm{d}^3 N_e}{\mathrm{d}\boldsymbol{p}^3} \equiv \langle \mathrm{vac;in}|a^{\mathrm{out}\dagger}_{\boldsymbol{p},s}a^{\mathrm{out}}_{\boldsymbol{p},s}|\mathrm{vac;in}\rangle = \sum_{s'}\int \mathrm{d}^3\boldsymbol{p}'\,\left|\beta_{\boldsymbol{p},s;\boldsymbol{p}',s'}\right|^2$$

Assume $\boldsymbol{E}_{\mathrm{s}}$ is so slow that it can be approximated as a static E-field

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⇒ one can evaluate $\beta_{\boldsymbol{p},s;\boldsymbol{p}',s'}$ **exactly !**

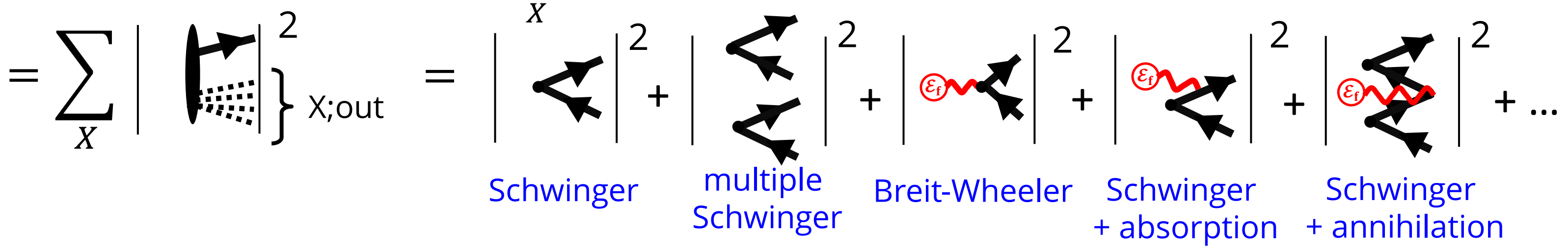
✔ Remarks

- **Directly computing VEV of # operator**

  ⇒ **inclusive quantity** that includes all the processes up to 1st order in $\mathcal{E}_{\mathrm{f}}$

  [Baltz, McLerran (2001)]

  $$\langle \mathrm{vac;in}|a^{\mathrm{out}\dagger}_{\boldsymbol{p},s}a^{\mathrm{out}}_{\boldsymbol{p},s}|\mathrm{vac;in}\rangle = \sum_X \left|\langle e_{\boldsymbol{p},s}X;\mathrm{out}|\mathrm{vac;in}\rangle\right|^2$$

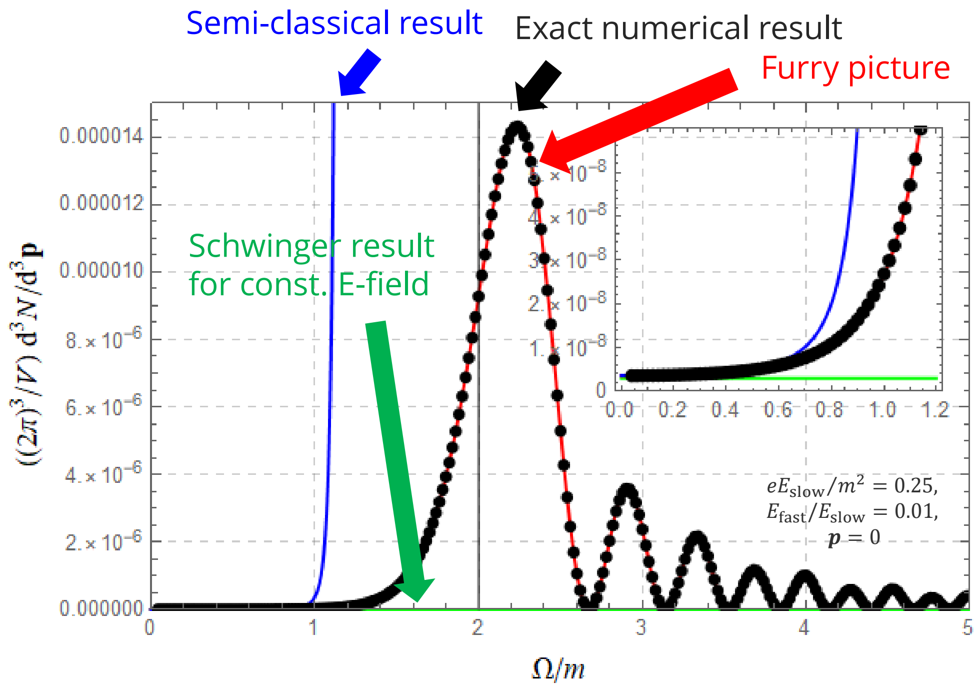


- **Production number ≠ Vacuum decay rate** [Cohen McGady (2008)] [Fukushima, Gelis, Lappi (2009)]
- **No approximation in evaluating $\beta_{\boldsymbol{p},s;\boldsymbol{p}',s'}$** within 0-th order WKB [Torgrimsson et al (2017)]

How good is the Furry picture ?

Result: Furry picture vs exact & semi-classical results

✔ **Case of $\boldsymbol{E} = (0, 0, \boldsymbol{E}_{\mathbf{s}} + \boldsymbol{\mathcal{E}}_{\mathbf{w}} \cos \boldsymbol{\Omega t})$**


Semi-classical result

Exact numerical result

Furry picture

Schwinger result
for const. E-field

✔ **Excellent agreement b/w Furry picture and exact results !**

- Furry picture is super powerful
- Physics lesson:

  The dynamically assisted Schwinger mechanism

  = One-photon process under E-field ⇒ **Perturbative effect is the essence**

Technically nice, but physically what are new ?

✔ **Furry-picture perturbation theory revealed:**

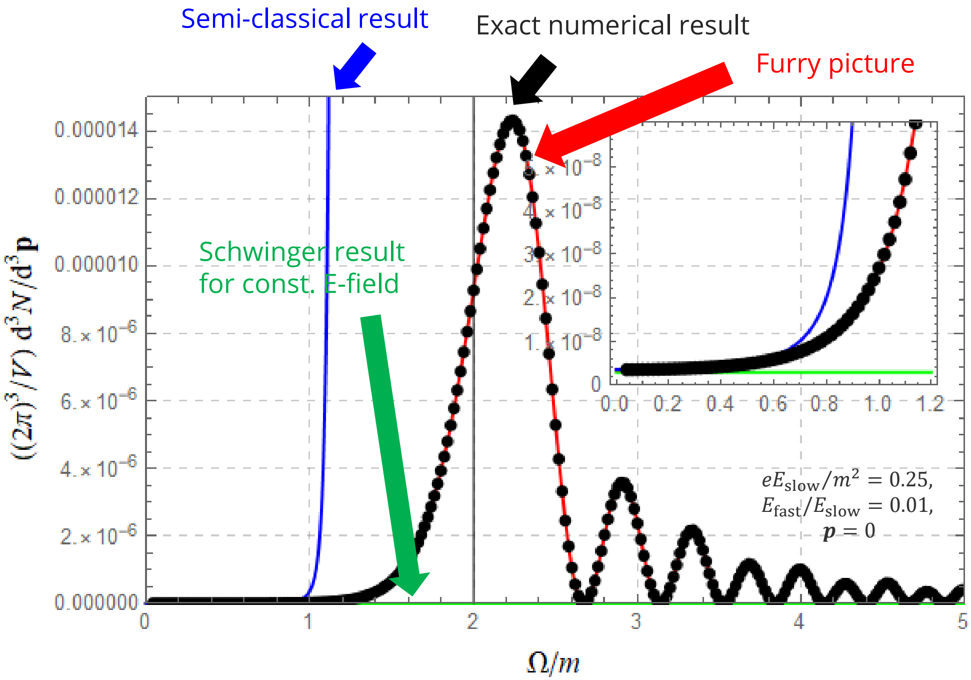
(1) Dynamically assisted Schwinger mech. beyond the semi-classical regime

(2) Spin production from the vacuum

(3) Effects spatial inhomogeneity

Result (1/3): Beyond the semi-classical regime

✔ **Case of $E = (0, 0, E_s + \mathcal{E}_w \cos \Omega t)$**



Semi-classical result

Exact numerical result

Furry picture

Schwinger result
for const. E-field

$eE_{\text{slow}}/m^2 = 0.25,$
$E_{\text{fast}}/E_{\text{slow}} = 0.01,$
$\boldsymbol{p} = 0$

"enhancement" is just a one aspect of the dynamically assisted Schwinger mech.

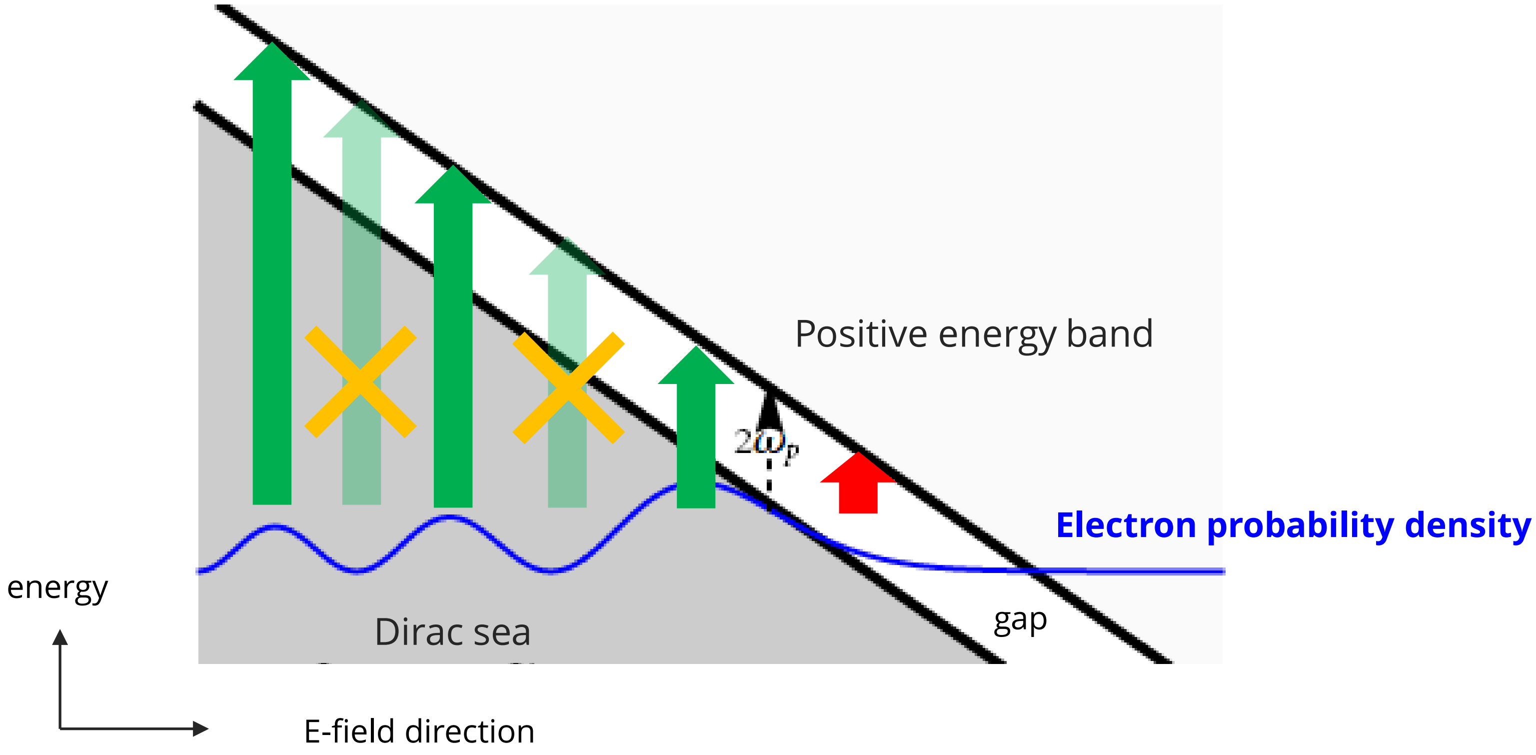
- shift of the one-photon peak $\Omega > 2m$ $\Rightarrow$ increase of the effective mass in E-field
- non-trivial oscillations above the one-photon peak

cf) Pondermotive energy
in AC E-field

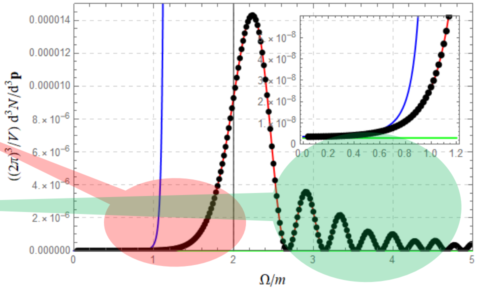
$\Leftarrow$ related to the "band structure" of the QED vacuum (*next slide*)

- can be interpreted as an analogue of the Franz-Keldysh effect in semi-conductor (*next next slide*)

Interpretation: “Band structure” of QED vacuum



- **Quantum tunneling ⇒ Enhancement**
- **Quantum reflection ⇒ Oscillation**



⇒ The spectrum of the dynamically assisted Schwinger mech.

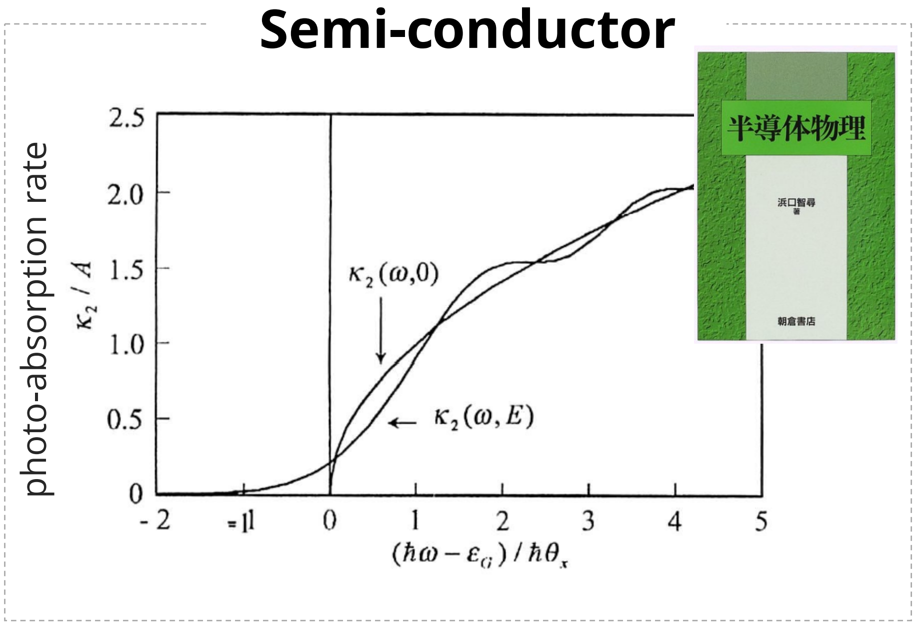
reflects the structure of the Dirac sea = QED vacuum

Analogue in semi-conductor: Franz-Keldysh effect

✔ **Enhancement & oscillation are analogous to Franz-Keldysh effect in semi-conductors**

[Franz (1958)] [Keldysh (1958)]

⇐ Apply strong slow E-field & a photon (~ weak fast E-field) onto a semi-cond., and measure photo-absorption rate


Semi-conductor

✔ The observables are essentially the same:

photo-absorption rate ∼ Im[1-loop action] ∼ total # of particles = int. of spectrum

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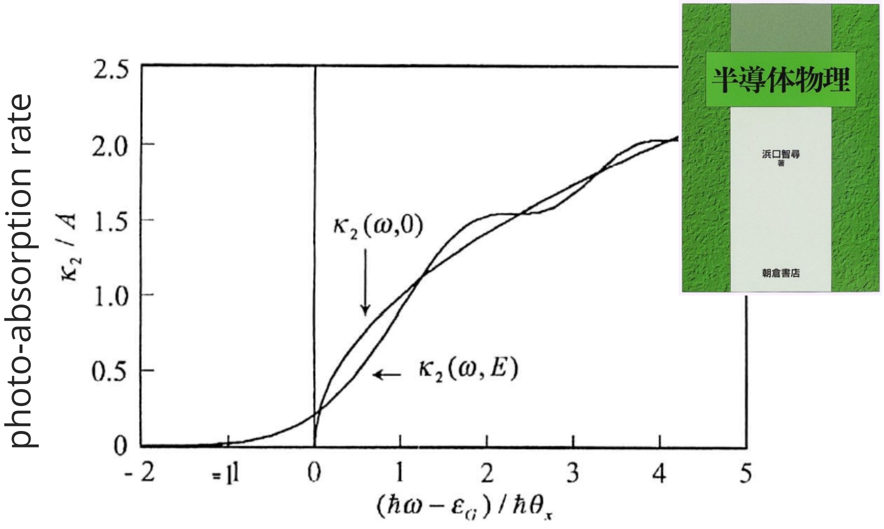
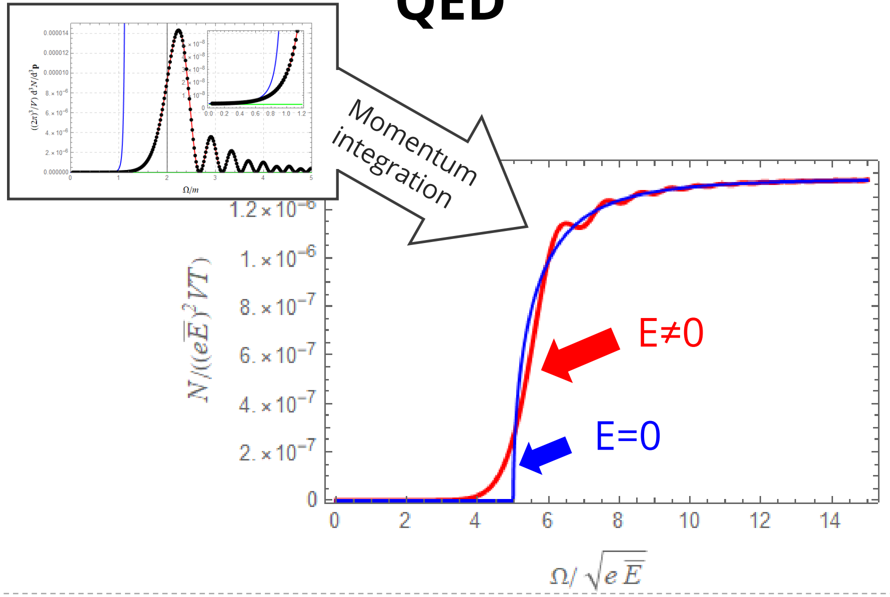


QED



✔ The observables are essentially the same:

photo-absorption rate ∼ Im[1-loop action] ∼ total # of particles = int. of spectrum

✔ The curves are the same !

⇒ the dynamically assisted Schwinger mech. (2008~) = Franz-Keldysh effect (1958~)

Technically nice, but physically what are new ?

✓ Furry-picture perturbation theory revealed:

(1) Dynamically assisted Schwinger mech. beyond the semi-classical regime

(2) Spin production from the vacuum

(3) Effects spatial inhomogeneity

Result (2/3): Spin production from the vacuum

✔ **The Schwinger mechanism is <u>usually</u> independent of spin**

(spin quantization axis ∝ E-field)

$$N_{e^\pm} = \frac{(eE)^2 VT}{(2\pi)^3} \times \exp\left[-\pi \frac{m^2}{eE}\right] \sim \exp[-\# \times (\text{gap height}) \times (\text{gap length})]$$

- linearly-polarized E-field does not couple to spin ⇒ gap is independent of spin
- true for the dynamically assisted Schwinger mechanism for $E = (0, 0, E_{\rm s} + \mathcal{E}_{\rm w}(t))$

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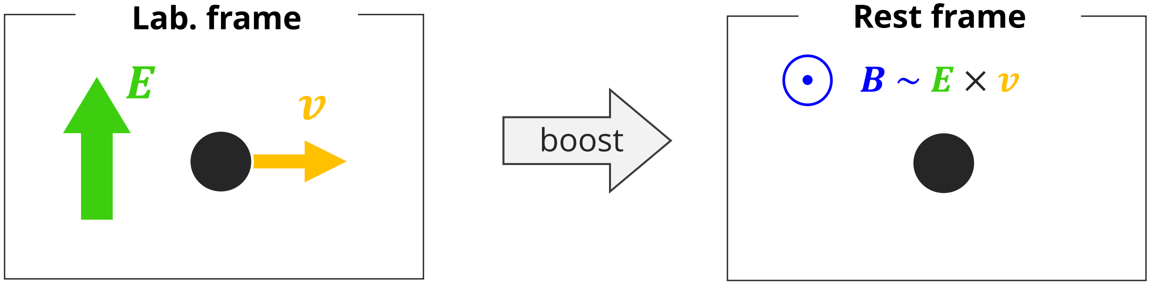
✔ **If E-field is circularly polarized (e.g., has transverse components $E = (\mathcal{E}_\mathrm{w}(t), 0, E_\mathrm{s})$), the Schwinger mechanism <u>can depend</u> on spin**

- intuitive reason: spin-orbit coupling ∝ $\boldsymbol{s} \cdot (\boldsymbol{v} \times \boldsymbol{E})$ [Foldy, Wouthuysen (1950)] [Tani (1951)]



⇒ spin dependence
due to Zeeman splitting

$$\Delta(\text{gap}) \sim \boldsymbol{s} \cdot \boldsymbol{B} \sim \boldsymbol{s} \cdot (\boldsymbol{v} \times \boldsymbol{E})$$

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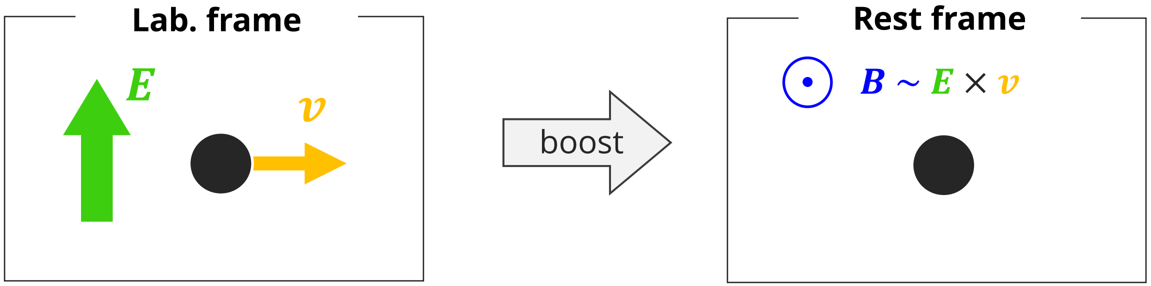


⇒ spin dependence
due to Zeeman splitting

$\Delta(\text{gap}) \sim \boldsymbol{s} \cdot \boldsymbol{B} \sim \boldsymbol{s} \cdot (\boldsymbol{v} \times \boldsymbol{E})$

✔ Technically, there was NO good analytical approach

$O(\hbar^0)$ terms will be important (=geometrical effects)

- naïve semi-classical formula can be applied only to linearly-polarized case [Takayoshi, Wu, Oka, (2021)] [Kitamura, Nagaosa, Morimoto, (2019)]

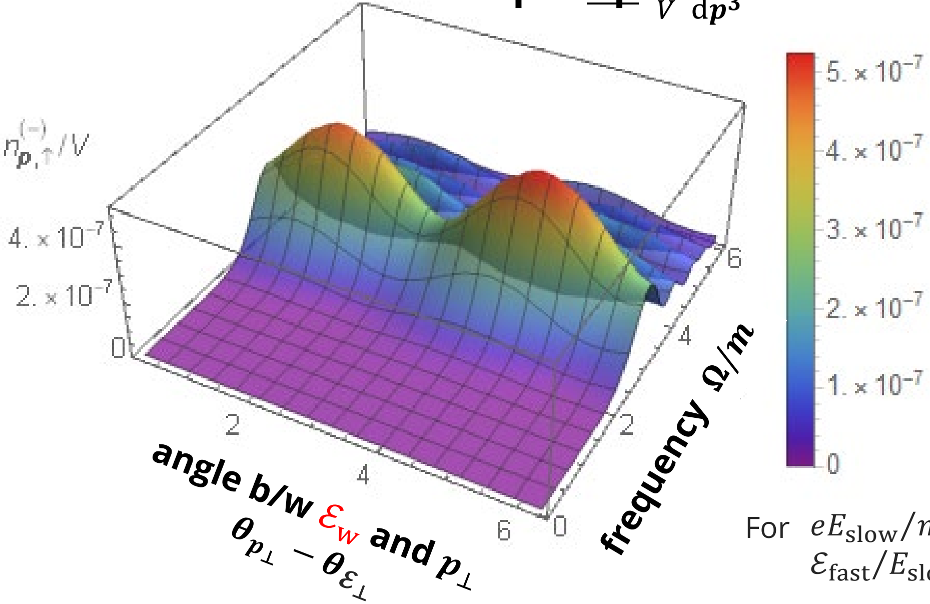
⇒ the perturbation theory in the Furry picture can deal with this problem !

Result (2/3): Spin production from the vacuum

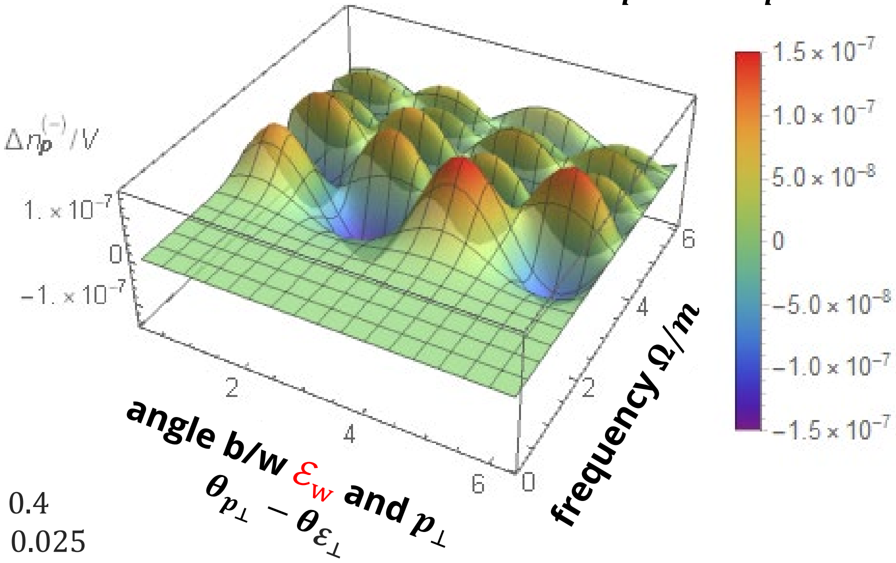
✔ Non-linearly-polarized case

$$E = \begin{pmatrix} \mathcal{E}_{\rm w} \cos \Omega t \times \cos \theta_{\mathcal{E}_\perp} \\ \mathcal{E}_{\rm w} \cos \Omega t \times \sin \theta_{\mathcal{E}_\perp} \\ E_{\rm s} \end{pmatrix}$$

momentum dist. of spin up $\frac{1}{V}\frac{{\rm d}^3 N_\uparrow}{{\rm d}\boldsymbol{p}^3}$



diff. b/w spin up and down $\frac{1}{V}\frac{{\rm d}^3 N_\uparrow}{{\rm d}\boldsymbol{p}^3} - \frac{1}{V}\frac{{\rm d}^3 N_\downarrow}{{\rm d}\boldsymbol{p}^3}$



For $eE_{\rm slow}/m^2 = 0.4$
$\mathcal{E}_{\rm fast}/E_{\rm slow} = 0.025$

- **Technical point:**

  Good agreement with numerics (though I haven't plotted here)

- **Physics lessons:**

  - Non-negligible (~10%) spin imbalance appears
  - Angular dependence due to spin-orbit coupling $\propto \boldsymbol{s} \cdot (\boldsymbol{p} \times \boldsymbol{E})$
  - No essential difference in enhancement & oscillation from the linearly-polarized case

Technically nice, but physically what are new ?

✔ Furry-picture perturbation theory revealed:

(1) Dynamically assisted Schwinger mech. beyond the semi-classical regime

(2) Spin production from the vacuum

(3) Effects spatial inhomogeneity

Result (3/3): Spatial inhomogeneity

✔ What if E-field has spatial inhomogeneity $\boldsymbol{E}(t) \to \boldsymbol{E}(t, \boldsymbol{x})$?

- No established analytical methods

[Bulanov et al. (2004)]

e.g.) locally-constant-field approximation: $N \sim V \, \exp\left[-\frac{\pi m^2}{eE}\right] \to \int \mathrm{d}^3\boldsymbol{x} \, \exp\left[-\frac{\pi m^2}{eE(t,x)}\right]$

⇒ NOT good even without spatial inhomogeneity…

- Numerical simulation is also heavy: 1+0+3 dim. → 1+3+3 dim.

t $\boldsymbol{x}$ $\boldsymbol{p}$

Result (3/3): Spatial inhomogeneity

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$t \nearrow$ $\boldsymbol{x} \nearrow$ $\boldsymbol{p} \nearrow$

✔ Perturbation theory in the Furry picture can be applied if:

$$\frac{\mathrm{d}^3 N_e}{\mathrm{d}\boldsymbol{p}^3} \equiv \langle \mathrm{vac;in} | a^{\mathrm{out}\dagger}_{\boldsymbol{p},s} a^{\mathrm{out}}_{\boldsymbol{p},s} | \mathrm{vac;in} \rangle = \sum_{s'} \int \mathrm{d}^3\boldsymbol{p}' \left| \beta_{\boldsymbol{p},s;\boldsymbol{p}',s'} \right|^2$$

where $$\beta_{\boldsymbol{p},s;\boldsymbol{p}',s'} = \int \mathrm{d}^3\boldsymbol{x}\ {}_{-}\psi^{(0)\mathrm{out}\dagger}_{\boldsymbol{p},s}\ {}_{+}\psi^{(0)\mathrm{in}}_{\boldsymbol{p}',s'} - i \int d^4x\ {}_{-}\bar{\psi}^{(0)\mathrm{out}}_{\boldsymbol{p},s}\, e\mathcal{A}_{\mathrm{w}}\ {}_{+}\psi^{(0)\mathrm{in}}_{\boldsymbol{p}',s'} + O(|e\mathcal{A}_{\mathrm{w}}|^2)$$

(1) the wave function ${}_{\pm}\psi^{(0)\mathrm{in}}_{\boldsymbol{p}',s'}$ under $e\boldsymbol{E}_{\mathrm{S}}$ is known

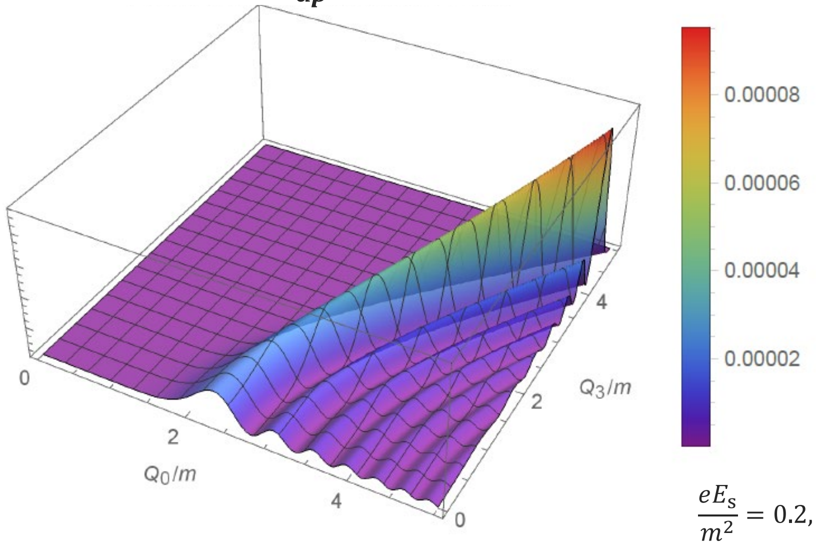
(2) $e\mathcal{A}_{\mathrm{w}}$ is sufficiently weak

[HT, unpublished]

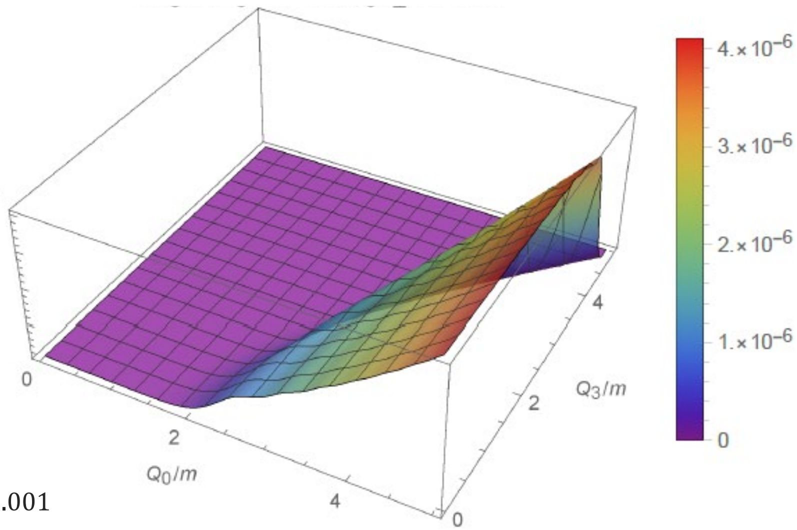
Result (3/3): Spatial inhomogeneity

✔ **Example:** $\boldsymbol{E} = (0, 0, E_{\rm s} + \mathcal{E}_{\rm w} \sin(Q_0 t - Q_3 z))$ (reduce to non-linear Breit-Wheeler for $Q_0 = Q_3$)

Spectrum $\frac{{\rm d}^3 N}{{\rm d}\boldsymbol{p}^3}$ at $\boldsymbol{p} = \boldsymbol{0}$



Total number N



$\frac{eE_{\rm s}}{m^2} = 0.2, \quad \frac{e\mathcal{E}_{\rm w}}{m^2} = 0.001$

- Essentially the same as the case without spatial inhomogeneity
  e.g.) enhancement below the threshold & oscillation above the threshold
- Role of spatial inhomogeneity
  = modify the thresholds by supplying momentum $Q_0 \sim 2m \to 2\sqrt{m^2 + Q_3^2}$

⇒ Time-dependence (i.e., energy injection) is the essence of the Schwinger mechanism

Spatial inhomogeneity may result in something interesting when the spatial inhomogeneity
(1) is strong and/or (2) has transverse components, for which magnetic effects (e.g., spin/chiral stuffs) should appear

Message of Part II

(1) The perturbation theory in the Furry picture is a very powerful analytical approach for the Schwinger mechanism with (space)time-dependent E-field

(2) The essence of the dynamically assisted Schwinger mech. is the pert. one-photon process

(3) The dynamically assisted Schwinger mechanism has rich physics other than simply enhancing the particle production (e.g., oscillation, mass shift, spin generation)

[HT, PRD (2019)] [Huang, HT, PRD (2019)]

Introduction

⇓

Part I:

Interplay b/w non-pert. & pert. production mechanisms

⇓

Part II:

Dynamically assisted Schwinger mechanism

⇓

Summary

Summary

I discussed the Schwinger mechanism under time-depending E-field:

Part I: Interplay b/w non-pert. & pert. production mechanisms

- The interplay is controlled by $\gamma \equiv \frac{m\Omega}{eE}$ (Keldysh parameter) and also by $\nu \equiv \frac{eE}{\Omega^2}$
- Semi-classical methods are invalid when $\nu \gtrsim 1$, where one-photon process dominates
- One-photon production is very efficient, compared to non-pert. tunneling

Part II: Dynamically assisted Schwinger mechanism

- **The perturbation theory in the Furry picture is a powerful analytical approach for the Schwinger mechanism with (space)time-dependent E-field**
- The essence of the dynamically assisted Schwinger mech. is the pert. one-photon process
- The dynamically assisted Schwinger mechanism has rich physics other than simply enhancing the particle production (e.g., oscillation, mass shift, spin generation)

- Interplay between perturbative & non-perturbative particle production:
  [HT, Fujii, Itakura, PRD (2014)] [HT, Fujimori, Misumi, Nitta, Sakai, JHEP (2021)]
- Dynamically assisted Schwinger mechanism and Furry-picture perturbation theory:
  [HT, PRD (2019)] [Huang, HT, PRD (2019)]
- Review on strong-field QED:
  [Fedotov, Ilderton, Karbstein, King, Seipt, HT, Torgrimsson, 2203.00019]

BACKUP

Validity of Furry-picture perturbation theory

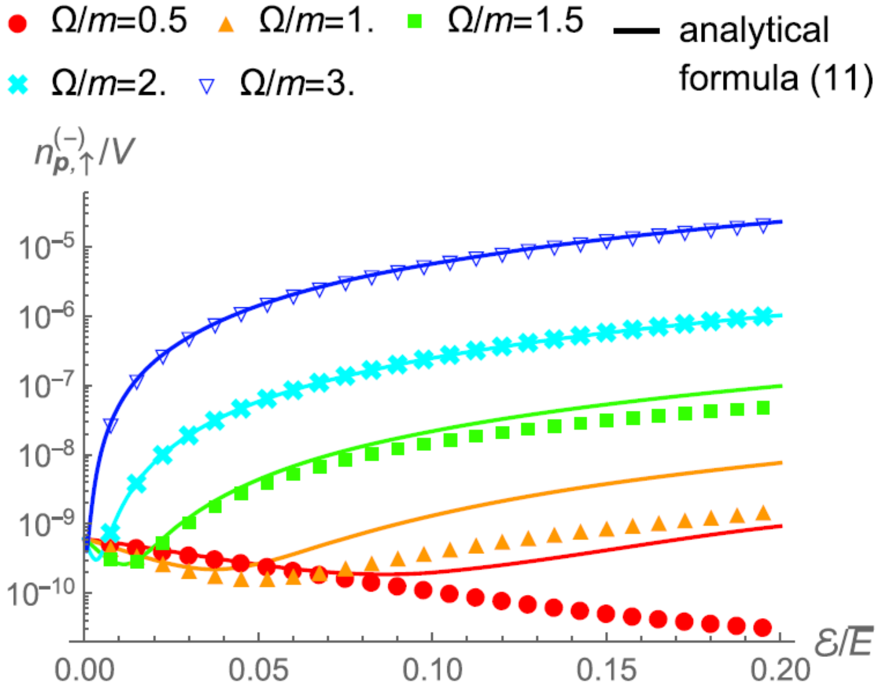


FIG. 9. A comparison between the numerical results (points) and the analytical results (lines) for the momentum distribution $n^{(-)}_{\boldsymbol{p},\uparrow}$ as a function of the strength of the perturbation $\mathcal{E}_\perp$ for several values of the frequency Ω. The parameters are the same as in Fig. 3, i.e., $e\bar{E}/m^2 = 0.4$, $\mathcal{E}_3/\bar{E} = 0$, $p_\perp/m = 1$, $p_3/m = 0$, $\phi = 1$, and $m\tau = 100$.

Furry-picture formula for parallel $E_\mathrm{s} \parallel \mathcal{E}_\mathrm{f}$

✔ **Analytical formula for $\boldsymbol{E} = (0,0,E_\mathrm{s} + \mathcal{E}_\mathrm{f}(t))$, with arbitrary time-dep.**

- is applicable even for very fast $\mathcal{E}_\mathrm{f}$ and reproduces numerics very well (show later)

$$\frac{\mathrm{d}^3 N_e}{\mathrm{d}\boldsymbol{p}^3} = \frac{V}{(2\pi)^3} \exp\left[-\frac{\pi(m^2+\boldsymbol{p}_\perp^2)}{eE_\mathrm{s}}\right]$$

$$\times \left| \underbrace{\boxed{1}}_{\text{Schwinger mech. by slow } E_\mathrm{s}} + \underbrace{\boxed{\frac{1}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{eE_\mathrm{s}} \int_0^\infty d\omega \frac{\tilde{\mathcal{E}}_\mathrm{f}(\omega)}{E_\mathrm{s}} \exp\left[-\frac{i}{4}\frac{\omega^2+4\omega p_\parallel}{eE_\mathrm{s}}\right] {}_1F_1\left(1-\frac{i}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{eE_\mathrm{s}}; 2; \frac{i}{2}\frac{\omega^2}{eE_\mathrm{s}}\right)}}_{\text{Dynamical assistance by fast } \mathcal{E}_\mathrm{f}} \right|^2$$

- describes interplay b/w Schwinger & one-photon process smoothly:

  - Slow limit $\omega/\sqrt{eE_\mathrm{s}} \ll 1$: 🟦 dominates ⇒ **Schwinger formula**

$$\frac{\mathrm{d}^3 N_e}{\mathrm{d}\boldsymbol{p}^3} \sim \frac{V}{(2\pi)^3} \exp\left[-\frac{\pi(m^2+\boldsymbol{p}_\perp^2)}{eE_\mathrm{s}}\right] \left|1 + \frac{\pi}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{eE_\mathrm{s}}\frac{\mathcal{E}_\mathrm{f}}{E_\mathrm{s}}\right|^2 \sim \frac{V}{(2\pi)^3} \exp\left[-\frac{\pi(m^2+\boldsymbol{p}_\perp^2)}{e(E_\mathrm{s}+\mathcal{E}_\mathrm{f})}\right]$$

  - Fast limit $\omega/\sqrt{eE_\mathrm{s}} \gg 1$: 🟥 dominates ⇒ **one-photon process**

$$\frac{\mathrm{d}^3 N_e}{\mathrm{d}\boldsymbol{p}^3} \sim \frac{V}{(2\pi)^3} \frac{1}{4}\frac{m^2+\boldsymbol{p}_\perp^2}{\omega_{\boldsymbol{p}}^2} \frac{|e\tilde{\mathcal{E}}_\mathrm{f}(2\omega_{\boldsymbol{p}})|^2}{\omega_{\boldsymbol{p}}^2}$$

Furry-picture formula for general $E_{\rm s} \nparallel \mathcal{E}_{\rm f}$

✔ **Generalization of the analytical formula for $E_{\rm s} \parallel \mathcal{E}_{\rm f} \to E_{\rm s} \nparallel \mathcal{E}_{\rm f}$**

$$\frac{{\rm d}^3 N_e}{{\rm d}\boldsymbol{p}^3} = \frac{V}{(2\pi)^3} \exp\left[-\frac{\pi(m^2+\boldsymbol{p}_\perp^2)}{eE_{\rm s}}\right]$$

$$\times \Bigg[\Bigg| \boxed{1} + \int_0^\infty d\omega\, {\rm e}^{-i\frac{\omega p_\parallel}{eE_{\rm s}}} \frac{1}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{eE_{\rm s}} \frac{\tilde{\boldsymbol{\mathcal{E}}}_{\rm f}(\omega)\cdot\boldsymbol{E}_{\rm s}}{E_{\rm s}^2} {\rm e}^{-i\frac{\omega^2}{4eE_{\rm s}}} {}_1F_1\left(1-\frac{i}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{eE_{\rm s}}; 2; \frac{i}{2}\frac{\omega^2}{eE_{\rm s}}\right)$$

$$+ i\int_0^\infty d\omega\, {\rm e}^{-i\frac{\omega p_\parallel}{eE_{\rm s}}} \frac{\tilde{\boldsymbol{\mathcal{E}}}_{\rm f}(\omega)\cdot\boldsymbol{p}_\perp}{E_{\rm s}\omega} {\rm Re}\left[{\rm e}^{-i\frac{\omega^2}{4eE_{\rm s}}} {}_1F_1\left(1-\frac{i}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{eE_{\rm s}}; 1; \frac{i}{2}\frac{\omega^2}{eE_{\rm s}}\right)\right]$$

$$+ s\times i\int_0^\infty d\omega\, {\rm e}^{-i\frac{\omega p_\parallel}{eE_{\rm s}}} \frac{(\tilde{\boldsymbol{\mathcal{E}}}_{\rm f}(\omega)\times\boldsymbol{p}_\perp)\cdot\boldsymbol{E}_{\rm s}}{E_{\rm s}^2\omega} {\rm Im}\left[{\rm e}^{-i\frac{\omega^2}{4eE_{\rm s}}} {}_1F_1\left(1-\frac{i}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{eE_{\rm s}}; 1; \frac{i}{2}\frac{\omega^2}{eE_{\rm s}}\right)\right] \Bigg|^2$$

$$+ \left| \int_0^\infty d\omega\, {\rm e}^{-i\frac{\omega p_\parallel}{eE_{\rm s}}} \frac{m}{\omega} \frac{\tilde{\mathcal{E}}_{\rm f}^x(\omega) + is\tilde{\mathcal{E}}_{\rm f}^y(\omega)}{E_{\rm s}} {\rm Im}\left[{\rm e}^{-i\frac{\omega^2}{4eE_{\rm s}}} {}_1F_1\left(1-\frac{i}{2}\frac{m^2+\boldsymbol{p}_\perp^2}{eE_{\rm s}}; 1; \frac{i}{2}\frac{\omega^2}{eE_{\rm s}}\right)\right] \right|^2 \Bigg]$$

Dynamical assistance by fast $\mathcal{E}_{\rm f}$

Schwinger mechanism by slow $E_{\rm s}$

- becomes complicated (green = new terms), but the basic structure is the same
- **spin-dependence appears even without magnetic fields** [Takayoshi, Wu, Oka (2020)]

  ∵ Dirac particle has a spin-orbit coupling $\boldsymbol{s}\cdot(\boldsymbol{p}\times\boldsymbol{\mathcal{E}})$

  [Foldy, Wouthuysen (1950)] [Tani (1951)]

- can be applied to rotating E-fields: $E = E_0(\cos(\Omega t), \sin(\Omega t), 0) \sim (\underset{E_{\rm s}}{E_0}, \underset{\mathcal{E}_{\rm f}}{E_0\Omega t}, 0)$

Numerical studies: [Blinne, Strobel (2015)] [Strobel, Xue (2015)] [Woller, Bauke, Keitel (2015)] [Kohlfurst (2019)]